The Problem with Continuous Casting

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Contents

Abstract

In this paper we will write a mathematical model for the continuous casting process in order to evaluate it's feasibility for producing steel sheets of varying thickness. We will initially discuss *moving boundary problems*, followed by *The Stefan Condition*. We will then discuss a simpler problem relating to a freezing boundary before moving on to the industrial problem. We will then discuss our results and evaluate the feasability of the approach.

Chapter 1

Introduction

1.1 Moving Boundary Problems

John Crank describes *Moving boundary problems* as

> associated with time-dependent problems, and the position of the boundary has to be determined as a function of time and space [1]

Also known as Stefan problems, these are problems where the (physical) boundary can change, or move, with time. These problems often occur in the context of phase changes - where a substance transitions from solid to liquid, or from liquid to gas - i.e. problems that involve:

- evaporation
- condensation
- melting
- solidification

In the context of these kinds of problems, an amount of heat energy is either required or released in order to transition from one phase to another: in the case of solidification / melting, the heat required to melt a substance is called the *latent heat of fusion*, and in the case of evaporation / condensation, the heat required to evaporate a substance is called the *latent heat of vaporisation*. We usually talk about the *specific latent heat* of a substance, denoted λ, which is the latent heat per unit mass of the substance.

1.2 The Stefan Condition

Stefan conditions deal with the boundary between two phases of a substance, such as ice and water, or water and steam. We will now derive a boundary condition to capture this evolving physical boundary.

Let us consider a (physical) boundary between ice and water as it freezes, and with an advancing frozen boundary. By applying conservation of energy we can say that the heat released by water as it freezes is equal to the the heat removed from the region by conduction. We consider that the boundary will advance δs in a period δt. Hence, the mass that solidifies in the time δt is:

$$\begin{aligned} m_{sol} &= \rho A(s(t+\delta t) - s(t)) \\ &= \rho A \delta s \end{aligned}$$

where ρ is the density of water, and A is the cross-sectional area of the boundary. Thus the heat released in the period δt is:

$$\begin{aligned} H_{sol} &= \lambda m_{sol} \\ &= \lambda \rho A \delta s \end{aligned}$$

where λ is the specific latent heat of fusion of water. We can assume that the water temperature is uniform, that the ice is colder than the water, and thus the heat released is conducted back through the ice. The amount of heat that flows across the boundary is therefore the heat flux multiplied by the cross-sectional area of the boundary. The heat flux is:

$$J = -k\frac{\partial u}{\partial x}$$

so the heat conducted back through the ice in the period δt is:

$$\begin{aligned} H_{con} &= -JA\delta t \\ &= k\frac{\partial u}{\partial x}(s(t), t)A\delta t \end{aligned}$$

where the negative value denotes the heat movement in the negative x direction. By equating both the heat released by solidification and the heat conducted back through the ice we get the following:

$$
\begin{aligned}
k\frac{\partial u}{\partial x}(s(t),t)A\delta t &= \lambda\rho A\delta s \\
k\frac{\partial u}{\partial x}(s(t),t) &= \lambda\rho\frac{\delta s}{\delta t}
\end{aligned}
$$

In the limit as $\delta t \to 0$, $\delta s \to 0$, and the equation becomes:

$$k\frac{\partial u}{\partial x}(s(t),t) = \lambda\rho\frac{ds}{dt}$$

1.3 Similarity Solution For The Heat Equation

We will now look for a general solution for the 1-D heat equation:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha\frac{\partial^2 u}{\partial x^2} \\
u(x,0) &= 0 \\
u(0,t) &= u_0 \\
u(\infty,t) &= 0
\end{aligned}
$$

To begin with, let us assume that the solution can be expressed as a function of both x and t in the following way:

$$u(x,t) = f(z)$$

where z is a function of both x and t:

$$z = \frac{x}{\sqrt{\alpha t}}$$

This will allow us to transform the equation into a simpler form. Starting with the first derivative of u with respect to t:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial f(z)}{\partial t} \\ &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial t} \\ &= f'(z) \cdot \left(\frac{-x\alpha}{2}\right) \cdot (\alpha t)^{-3/2}\end{aligned}$$

followed by the first derivative of u with respect to x:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial f(z)}{\partial x} \\ &= \frac{\partial f(z)}{\partial z} \cdot \frac{\partial z}{\partial x} \\ &= \frac{f'(z)}{\sqrt{\alpha t}}\end{aligned}$$

and finally the second derivative of u with respect to x:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{f'(z)}{\sqrt{\alpha t}}\right) \\ &= \frac{1}{\sqrt{\alpha t}} \cdot \frac{\partial f'(z)}{\partial x} \\ &= \frac{1}{\sqrt{\alpha t}} \cdot f''(z) \cdot \frac{\partial z}{\partial x} \\ &= \frac{1}{\sqrt{\alpha t}} \cdot f''(z) \cdot \frac{1}{\sqrt{\alpha t}} \\ &= \frac{f''(z)}{\alpha t}\end{aligned}$$

Substituting the above into the heat equation we get:

$$\begin{aligned}
f'(z) \cdot \left(\frac{-x\alpha}{2}\right) \cdot (\alpha t)^{-3/2} &= \alpha \cdot \frac{f''(z)}{\alpha t} \\
-f'(z) \cdot \frac{x}{2\sqrt{\alpha t}} &= f''(z) \\
-\frac{z}{2} f'(z) &= f''(z)
\end{aligned}$$

So we have reduced the heat equation to an ordinary differential equation:

$$f''(z) + \frac{z}{2} f'(z) = 0$$

which can be easily solved for $f'(z)$, followed by $f(x)$, as follows:

$$\begin{aligned}
\frac{df'}{dz} &= -\frac{z}{2} f' \\
\int \frac{df'}{f'} &= -\int \frac{z}{2} dz \\
ln(f') &= -\frac{z^2}{4} + C \\
f'(z) &= c_1 e^{-z^2/4} \\
f(z) &= c_1 \int_0^z e^{-w^2/4} dw + c_2
\end{aligned}$$

we will make the following substitution for clarity:

$$\begin{aligned}
v &= \frac{w}{2} \\
dv &= \frac{1}{2} dw
\end{aligned}$$

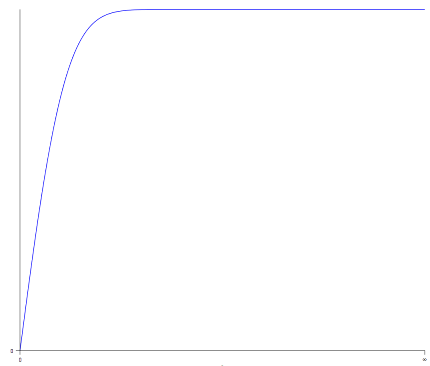

Figure 1.1: The error function, $\text{erf}(x), 0 < x < \infty$

which results in:

$$f(z) = 2c_1 \int_0^{z/2} e^{-v^2} dv + c_2$$

which is recognizable as the error function (see figure 1.1):

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$$

so our solution can be rewritten as:

$$\begin{aligned} f(z) &= \sqrt{\pi} c_1 \, \text{erf}\left(\frac{z}{2}\right) + c_2 \\ &= C_1 \, \text{erf}\left(\frac{z}{2}\right) + C_2 \end{aligned}$$

and our final solution is:

$$u(x,t) = C_1 \, \text{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) + C_2$$

Chapter 2

The Freezing Problem

2.1 The One-Phase Stefan Problem

To begin lets take a simple problem - that of a semi-infinite region $0 < x < \infty$ within which liquid freezes. Let us assume that the liquid freezes from left to right, with a solidification front $x = s(t)$. Let $u(x,t)$ be the temperature of the solidified water (ice). Assume the initial temperature of the liquid is at 0°C, and the temperature on the left boundary is kept at -1°C (just below freezing). We can describe our system with the following equations:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < s(t), \qquad t > 0 \\
u(0,t) &= -1 \\
u(s(t),t) &= 0
\end{aligned}
$$

where α is the diffusivity. As we don't know the nature of $s(t)$ - the solidification front - we need another condition to help us find it. This boundary condition, a *Stefan Condition*, we have discussed in detail in the previous chapter. So we add two more conditions to our system of equations:

$$
\begin{aligned}
k\frac{\partial u}{\partial x}(s(t),t) &= \lambda \rho \frac{ds}{dt} \\
s(0) &= 0
\end{aligned}
$$

where λ in this case is the specific latent heat of fusion for water. Our similarity solution to the heat equation we found to be:

$$u(x,t) = C_1 \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha t}}\right) + C_2$$

We need to find C_1 and C_2. To do this, lets first assume all constants are equal to 1:

$$\alpha = \lambda = \rho = k = 1$$

By using our boundary condition $u(s(t), t) = 0$ we get:

$$\begin{aligned} C_1 \operatorname{erf}\left(\frac{s(t)}{2\sqrt{t}}\right) + C_2 &= 0 \\ \operatorname{erf}\left(\frac{s(t)}{2\sqrt{t}}\right) &= -\frac{C_2}{C_1} \end{aligned}$$

which implies that $\frac{s(t)}{2\sqrt{t}}$ must be a constant.

$$\begin{aligned} \frac{s(t)}{2\sqrt{t}} &= m \\ s(t) &= 2m\sqrt{t} \end{aligned}$$

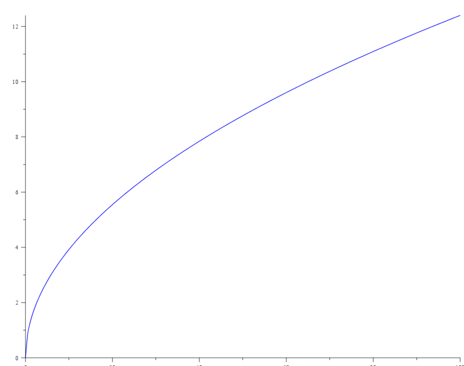

Figure 2.1: The moving boundary $s(t)$

Using our other boundary condition $u(0,t) = -1$, and with $\mathrm{erf}(0) = 0$, we get for C_2:

$$\begin{aligned} C_1 \,\mathrm{erf}(0) + C_2 &= -1 \\ C_2 &= -1 \end{aligned}$$

solving for C_1:

$$\begin{aligned} C_1 \,\mathrm{erf}(m) - 1 &= 0 \\ C_1 &= \frac{1}{\mathrm{erf}(m)} \end{aligned}$$

putting C_1 and C_2 back into our solution for $u(x,t)$:

$$u(x,t) = \frac{\mathrm{erf}(x/2\sqrt{t})}{\mathrm{erf}(m)} - 1$$

Now we substitute our solution for $u(x,t)$ into the Stefan condition in order to find the unknown m:

$$\frac{\partial u}{\partial x}(s(t),t) = \frac{ds}{dt}$$

beginning with the LHS:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{\mathrm{erf}(m)} \cdot \frac{\partial}{\partial x}\left[\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)\right] \\ &= \frac{1}{\mathrm{erf}(m)} \cdot \frac{1}{2\sqrt{t}} \cdot \frac{2}{\sqrt{\pi}} \cdot e^{-(x/2\sqrt{t})^2} \\ &= \frac{1}{\mathrm{erf}(m)} \cdot \frac{1}{\sqrt{\pi t}} \cdot e^{-(x/2\sqrt{t})^2} \end{aligned}$$

evaluated at $x = s(t)$:

$$
\begin{aligned}
\frac{\partial u}{\partial x}(s(t),t) &= \frac{1}{\mathrm{erf}(m)} \cdot \frac{1}{\sqrt{\pi t}} \cdot e^{-\left(s(t)/2\sqrt{t}\right)^2} \\
&= \frac{1}{\mathrm{erf}(m)} \cdot \frac{1}{\sqrt{\pi t}} \cdot e^{-m^2}
\end{aligned}
$$

moving to the RHS:

$$
\begin{aligned}
s(t) &= 2m\sqrt{t} \\
\frac{ds}{dt} &= \frac{m}{\sqrt{t}}
\end{aligned}
$$

and finally equating the LHS and the RHS:

$$
\begin{aligned}
\frac{1}{\mathrm{erf}(m)} \cdot \frac{1}{\sqrt{\pi t}} \cdot e^{-m^2} &= \frac{m}{\sqrt{t}} \\
m \cdot \mathrm{erf}(m) \cdot e^{m^2} &= \frac{1}{\sqrt{\pi}}
\end{aligned}
$$

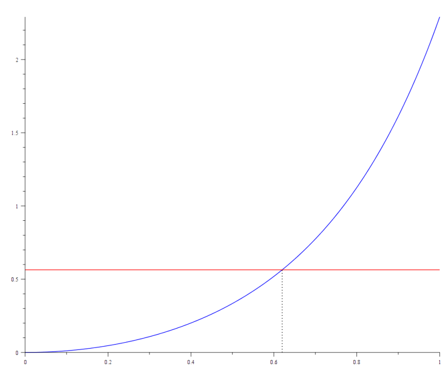

Figure 2.2: Solving for m graphically

which we can solve graphically (see figure 2.2) or numerically. Using Maple and the *fsolve* function we find the numerical solution for m to be 0.6201 to 4 decimal places.

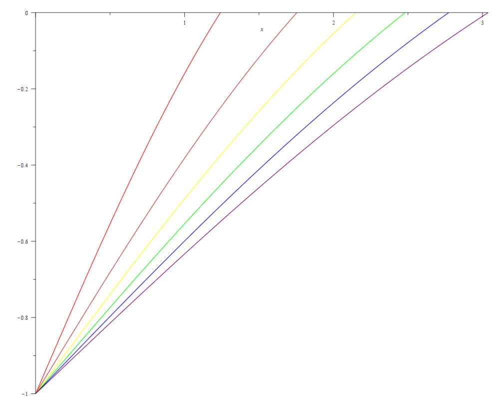

Figure 2.3: Temperature (u) for various different values of x

In figure 2.3 we see a plot of the temperature u for values of x ranging from 1 to 6, and in figure 2.4 we see a contour plot of the temperature for differing values of x and t, with temperature ranging from blue (colder) to grey (warmer).

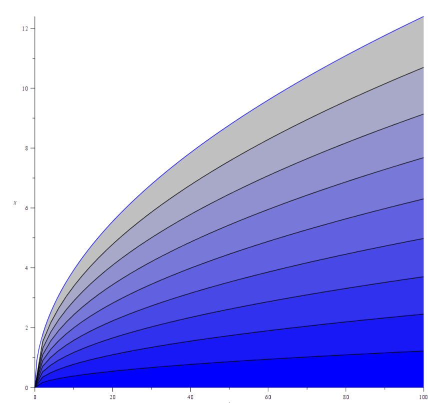

Figure 2.4: A contour plot of Temperature (u) versus x and t

2.2 The Pseudo-Steady-State Approximation Method

The speudo-steady-state approximation method is a solution obtained under the assumption that the time derivative in the heat equation is negligible, and hence can be set to 0. To re-state our freezing problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < s(t), \qquad t > 0 \\
u(0,t) &= -1 \\
u(s(t),t) &= 0
\end{aligned}$$

with Stefan Condition:

$$\begin{aligned}
\frac{\partial u}{\partial x}(s(t),t) &= \frac{ds}{dt} \\
s(0) &= 0
\end{aligned}$$

Under the assumption that the time derivative can be set to 0, the heat equation reduces to:

$$\frac{\partial^2 u}{\partial x^2} = 0$$

which can be solved easily:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= 0 \\
\frac{\partial u}{\partial x} &= c_1(t) \\
u &= c_1(t)x + c_2(t)
\end{aligned}$$

and the constants of integration, c_1 and c_2 can be obtained readily using the boundary conditions. Our first boundary condition, $u(0,t) = -1$, gives us:

$$\begin{aligned}
c_1(t) \cdot 0 + c_2(t) &= -1 \\
c_2(t) &= -1
\end{aligned}$$

and our second boundary condition, $u(s(t), t) = 0$, gives us:

$$\begin{aligned} c_1(t) \cdot s(t) - 1 &= 0 \\ c_1(t) &= \frac{1}{s(t)} \end{aligned}$$

so our general solution for u is:

$$u(x, t) = \frac{x}{s(t)} - 1$$

it's first derivative with respect to x is:

$$\frac{\partial u}{\partial x} = \frac{1}{s(t)}$$

and making use of the first Stefan Condition:

$$\begin{aligned} \frac{ds}{dt} &= \frac{1}{s} \\ s \cdot ds &= dt \\ \int s \cdot ds &= \int dt \\ \frac{s^2}{2} &= t + c \\ s &= \sqrt{2t} + C \end{aligned}$$

and using the second Stefan Condition, $s(0) = 0$, we get:

$$\begin{aligned} \sqrt{2 \cdot 0} + C &= 0 \\ C &= 0 \end{aligned}$$

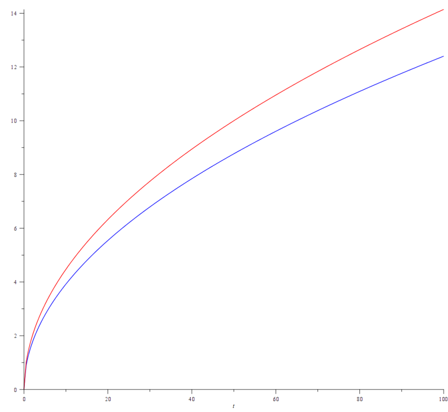

Figure 2.5: A plot of the exact (blue) versus approximate (red) solutions for s

which gives us:

$$
\begin{aligned}
s(t) &= \sqrt{2t} \\
u(x,t) &= \frac{x}{\sqrt{2t}} - 1
\end{aligned}
$$

So how does this approximation compare with the exact solution previously discussed? Looking at figure 2.5, which plots the exact versus approximate solution for s, we can see they are reasonably close.

But what about u? In figure 2.6, which plots the exact versus approximate solution for u for a number of different values of time, we can see that they are also

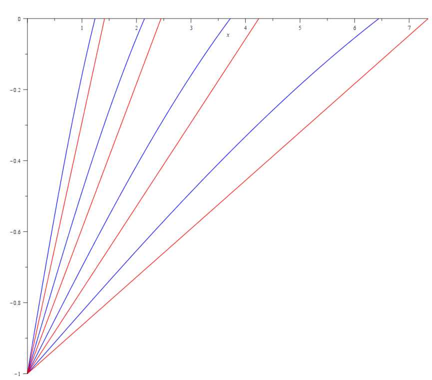

Figure 2.6: A plot of the exact (blue) versus approximate (red) solutions for u

reasonably close. In both cases the approximation is not perfect, but is certainly an acceptable approximation for drawing conclusions about the behaviour of the modelled system.

2.3 The Two-Phase Stefan Problem

In the previous approach we solved for the moving boundary and the temperature of the solid region behind the moving boundary. We were not concerned with the temperature of the water.

We will now consider a Two-Phase Stefan Problem - one where the temperatures of both phases are considered. We will reformulate our problem slightly:

$$\begin{aligned}
\frac{\partial u_S}{\partial t} &= \alpha_S \frac{\partial^2 u_S}{\partial x^2}, \qquad 0 < x < s(t), \qquad t > 0 \\
\frac{\partial u_L}{\partial t} &= \alpha_L \frac{\partial^2 u_L}{\partial x^2}, \qquad s(t) < x < \infty, \qquad t > 0 \\
u_S(0,t) &= u_1 \\
u_S(s(t),t) &= 0 \\
u_L(s(t),t) &= 0 \\
u_L(x,0) &= u_0
\end{aligned}$$

where u_S is the temperature in the solid region, u_L is the temperature in the liquid region, u_1 is the temperature on the left boundary, u_0 is the initial temperature of the liquid phase, and α_S, α_L are the diffusivities of the solid and liquid regions respectively.

The Stefan condition is as follows:

$$-k_L \frac{\partial u_L}{\partial x}(s(t),t) + k_S \frac{\partial u_S}{\partial x}(s(t),t) = \rho\lambda \frac{ds}{dt}$$

We will use the similarity solution for the heat equation:

$$\begin{aligned}
u_S(x,t) &= C_1 \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_S t}}\right) + C_2 \\
u_L(x,t) &= C_3 \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_L t}}\right) + C_4
\end{aligned}$$

By using our boundary condition $u_S(s(t), t) = 0$ we get:

$$\begin{aligned} C_1 \operatorname{erf}\left(\frac{s(t)}{\sqrt{4\alpha_S t}}\right) + C_2 &= 0 \\ \operatorname{erf}\left(\frac{s(t)}{\sqrt{4\alpha_S t}}\right) &= -\frac{C_2}{C_1} \end{aligned}$$

which implies that $\frac{s(t)}{\sqrt{4\alpha_S t}}$ must be a constant.

$$\begin{aligned} \frac{s(t)}{\sqrt{4\alpha_S t}} &= m \\ s(t) &= m\sqrt{4\alpha_S t} \end{aligned}$$

We need to find C_1, C_2, C_3, and C_4. Starting with u_S and using our boundary condition $u_S(0, t) = u_1$:

$$\begin{aligned} C_1 \operatorname{erf}\left(\frac{0}{\sqrt{4\alpha_S t}}\right) + C_2 &= u_1 \\ C_1 \operatorname{erf}(0) + C_2 &= u_1 \\ C_1 \cdot 0 + C_2 &= u_1 \\ C_2 &= u_1 \end{aligned}$$

returning to our boundary condition $u_S(s(t), t) = 0$ we get:

$$\begin{aligned} C_1 \operatorname{erf}(m) + C_2 &= 0 \\ C_1 &= -\frac{C_2}{\operatorname{erf}(m)} \\ &= -\frac{u_1}{\operatorname{erf}(m)} \end{aligned}$$

so u_S can be written:

$$\begin{aligned} u_S(x,t) &= u_1 - \frac{u_1}{\operatorname{erf}(m)} \cdot \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_S t}}\right) \\ &= u_1\left[1 - \frac{\operatorname{erf}(x/\sqrt{4\alpha_S t})}{\operatorname{erf}(m)}\right] \end{aligned}$$

Next u_L, and by using our boundary condition $u_L(x,0) = u_0$ we get:

$$\begin{aligned} C_3 \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_L 0}}\right) + C_4 &= u_0 \\ C_3 \operatorname{erf}(\infty) + C_4 &= u_0 \\ C_3 \cdot 1 + C_4 &= u_0 \\ C_3 + C_4 &= u_0 \end{aligned}$$

and using our boundary condition $u_L(s(t),t) = 0$ we get:

$$\begin{aligned} C_3 \operatorname{erf}\left(\frac{s(t)}{\sqrt{4\alpha_L t}}\right) + C_4 &= 0 \\ C_3 \operatorname{erf}\left(\frac{m\sqrt{4\alpha_S t}}{\sqrt{4\alpha_L t}}\right) + C_4 &= 0 \\ C_3 \operatorname{erf}\left(m\sqrt{\frac{\alpha_S}{\alpha_L}}\right) + C_4 &= 0 \end{aligned}$$

and by using the fact that $C_3 + C_4 = u_0$ we can find C_3:

$$
\begin{aligned}
C_3 \operatorname{erf}\left(m\sqrt{\frac{\alpha_S}{\alpha_L}}\right) + C_4 &= 0 \\
C_3 \operatorname{erf}\left(m\sqrt{\frac{\alpha_S}{\alpha_L}}\right) + u_0 - C_3 &= 0 \\
u_0 &= C_3 - C_3 \operatorname{erf}\left(m\sqrt{\frac{\alpha_S}{\alpha_L}}\right) \\
&= C_3\left[1 - \operatorname{erf}\left(m\sqrt{\frac{\alpha_S}{\alpha_L}}\right)\right] \\
C_3 &= \frac{u_0}{1 - \operatorname{erf}\left(m\sqrt{\alpha_S/\alpha_L}\right)}
\end{aligned}
$$

from which we can find C_4:

$$
\begin{aligned}
C_4 &= -C_3 \operatorname{erf}\left(m\sqrt{\frac{\alpha_S}{\alpha_L}}\right) \\
C_4 &= -\frac{u_0 \operatorname{erf}(m\sqrt{\alpha_S/\alpha_L})}{1 - \operatorname{erf}(m\sqrt{\alpha_S/\alpha_L})}
\end{aligned}
$$

so u_L can be written:

$$
\begin{aligned}
u_L(x,t) &= \frac{u_0 \operatorname{erf}(x/\sqrt{4\alpha_L t})}{1 - \operatorname{erf}(m\sqrt{\alpha_S/\alpha_L})} - \frac{u_0 \operatorname{erf}(m\sqrt{\alpha_S/\alpha_L})}{1 - \operatorname{erf}(m\sqrt{\alpha_S/\alpha_L})} \\
&= u_0\left[1 - \frac{1 - \operatorname{erf}(x/\sqrt{4\alpha_L t})}{1 - \operatorname{erf}(m\sqrt{\alpha_S/\alpha_L})}\right]
\end{aligned}
$$

Now we use the Stefan condition in order to find the unknown m. We first calculate the following derivatives:

$$\begin{aligned}
\frac{\partial u_S}{\partial x}(s(t),t) &= \frac{u_1 e^{-m^2}}{\mathrm{erf}(m)\sqrt{\pi\alpha_S t}} \\
\frac{\partial u_L}{\partial x}(s(t),t) &= \frac{u_0 e^{-(m\sqrt{\alpha_S/\alpha_L})^2}}{(1-\mathrm{erf}(m\sqrt{\alpha_S/\alpha_L}))\sqrt{\pi\alpha_L t}} \\
\frac{ds}{dt} &= m\sqrt{\frac{\alpha_S}{t}}
\end{aligned}$$

and then we substitute these into the Stefan condition, which gives us:

$$-k_L\left(\frac{u_0 e^{-(m\sqrt{\alpha_S/\alpha_L})^2}}{(1-\mathrm{erf}(m\sqrt{\alpha_S/\alpha_L}))\sqrt{\pi\alpha_L t}}\right) - k_S\left(\frac{u_1 e^{-m^2}}{\mathrm{erf}(m)\sqrt{\pi\alpha_S t}}\right) = \rho\lambda m\sqrt{\frac{\alpha_S}{t}}$$

which can be solved numerically using maple.

Chapter 3

The Continuous Casting Problem

3.1 Description

We are interested in modelling a method that produces steel sheets by pouring molten steel onto a water-cooled rotating drum. As the molten steel hits the cooled drum it begins to cool and solidify. As it cools a "puddle" is formed from where the molten steel first comes into contact with the drum to where it has fully solidified. In order for this process to be successful, the puddle must disappear before the sheet is removed from the drum.

The thickness of the sheets would be controlled by varying the rate of flow of the molten steel. If the steel sheets cool and solidify quickly enough they can be removed and further processed.

We will now attempt to model this problem. Our approach will be to try to find the length of the puddle. For a drum rotating with speed V the length of the puddle is simply:

$$\ell = Vt_h$$

where t_h is the time taken for the molten steel to solidify to a thickness h. What we are really interested in is the quantiy t_h. To find this time we can take a similar approach to the freezing problem previously discussed and treat this problem as a one-dimensional problem where the solidification boundary moves from the surface of the cooled drum to the extent of the molten metal.

3.2 Solution

Let $u_1(x,t)$ denote the temperature of the drum, and let $u_2(x,t)$ denote the temperature of the solidified steel. And let $x = s(t)$ be the moving boundary between the solidified steel and the molten steel.

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= \alpha_1 \frac{\partial^2 u_1}{\partial x^2}, \qquad -\infty < x < 0 \\ \frac{\partial u_2}{\partial t} &= \alpha_2 \frac{\partial^2 u_2}{\partial x^2}, \qquad 0 < x < s(t)\end{aligned}$$

where α_1 and α_2 are the diffusivities of the drum material and the solidified steel respectively. Our boundary conditions are:

$$\begin{aligned}u_1(-\infty,t) &= u_d \\ u_1(0,t) &= u_2(0,t) \\ u_2(s(t),t) &= u_f\end{aligned}$$

where u_d is the temperature at the core of the drum, u_f is the temperature that molten steel solidifies. Our continuity of flux condition is:

$$-k_1 \frac{\partial u_1}{\partial x}(0,t) = -k_2 \frac{\partial u_2}{\partial x}(0,t)$$

which basically states that the heat flux in the drum is the same for the metal. k_1 and k_2 are the thermal conductivities for the drum material and the steel respectively. And finally, our Stefan condition is:

$$k_2 \frac{\partial u_2}{\partial x}(s(t),t) = \rho_2 \lambda \frac{ds}{dt}$$

In order to turn this problem into two seperate problems that we can solve independantly, we introduce a new value, U, which is the temperature at the point of contact between the drum and the steel:

$$u_1(0,t) = u_2(0,t) = U$$

So we now have two problems. For the drum:

$$\begin{aligned}
\frac{\partial u_1}{\partial t} &= \alpha_1 \frac{\partial^2 u_1}{\partial x^2}, \qquad -\infty < x < 0 \\
u_1(-\infty, t) &= u_d \\
u_1(0, t) &= U
\end{aligned}$$

and for the steel:

$$\begin{aligned}
\frac{\partial u_2}{\partial t} &= \alpha_2 \frac{\partial^2 u_2}{\partial x^2}, \qquad 0 < x < s(t) \\
u_2(s(t), t) &= u_f \\
u_2(0, t) &= U
\end{aligned}$$

To solve this problem we will once again use the similarity solution for the heat equation:

$$\begin{aligned}
u_1(x,t) &= C_1 \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_1 t}}\right) + C_2 \\
u_2(x,t) &= C_3 \operatorname{erf}\left(\frac{x}{\sqrt{4\alpha_2 t}}\right) + C_4
\end{aligned}$$

By using our boundary condition $u_2(s(t), t) = u_f$ we see that:

$$\begin{aligned}
C_3 \operatorname{erf}\left(\frac{s(t)}{\sqrt{4\alpha_2 t}}\right) + C_4 &= u_f \\
\operatorname{erf}\left(\frac{s(t)}{\sqrt{4\alpha_2 t}}\right) &= \frac{u_f - C_4}{C_3}
\end{aligned}$$

which once again implies that $\frac{s(t)}{\sqrt{4\alpha_2 t}}$ must be a constant.

$$\begin{aligned}
\frac{s(t)}{\sqrt{4\alpha_2 t}} &= m \\
s(t) &= m\sqrt{4\alpha_2 t}
\end{aligned}$$

We need to find C_1, C_2, C_3, and C_4. Starting with u_1, and by using our boundary condition $u_1(0,t) = U$ we get:

$$\begin{aligned}
C_1 \operatorname{erf}(0) + C_2 &= U \\
C_2 &= U
\end{aligned}$$

and by using our boundary condition $u_1(-\infty, t) = u_d$ we get:

$$\begin{aligned}
C_1 \operatorname{erf}(-\infty) + C_2 &= u_d \\
-C_1 + C_2 &= u_d \\
C_1 &= C_2 - u_d \\
&= U - u_d
\end{aligned}$$

so now we have:

$$u_1(x,t) = U + (U - u_d) \operatorname{erf}(x/\sqrt{4\alpha_1 t})$$

Moving on to u_2, and by using our boundary condition $u_2(0,t) = U$ we get:

$$\begin{aligned}
C_3 \operatorname{erf}(0) + C_4 &= U \\
C_4 &= U
\end{aligned}$$

and again by using our boundary condition $u_2(s(t), t) = u_f$ we get:

$$
\begin{aligned}
C_3 \operatorname{erf}(m) + C_4 &= u_f \\
C_3 &= \frac{u_f - C_4}{\operatorname{erf}(m)} \\
&= \frac{u_f - U}{\operatorname{erf}(m)}
\end{aligned}
$$

so now we have:

$$
u_2(x,t) = U + (u_f - U)\frac{\operatorname{erf}(x/\sqrt{4\alpha_2 t})}{\operatorname{erf}(m)}
$$

All that remains is to solve for U and m. We first calculate the following derivatives:

$$
\begin{aligned}
\frac{\partial u_1}{\partial x} &= \frac{(U - u_d)e^{-(x/\sqrt{4\alpha_1 t})^2}}{\sqrt{\pi\alpha_1 t}} \\
\frac{\partial u_2}{\partial x} &= \frac{(u_f - U)e^{-(x/\sqrt{4\alpha_2 t})^2}}{\operatorname{erf}(m)\sqrt{\pi\alpha_2 t}} \\
\frac{ds}{dt} &= m\sqrt{\frac{\alpha_2}{t}}
\end{aligned}
$$

and then we substitute into the remaining boundary conditions, starting with the continuity of flux:

$$
\begin{aligned}
-k_1 \frac{U - u_d}{\sqrt{\pi\alpha_1 t}} &= -k_2 \frac{u_f - U}{\operatorname{erf}(m)\sqrt{\pi\alpha_2 t}} \\
U - u_d &= \frac{k_2}{k_1}\sqrt{\frac{\alpha_1}{\alpha_2}}\frac{u_f - U}{\operatorname{erf}(m)} \\
&= \beta\frac{u_f - U}{\operatorname{erf}(m)} \qquad \text{where} \qquad \beta = \frac{k_2}{k_1}\sqrt{\frac{\alpha_1}{\alpha_2}} \\
U \operatorname{erf}(m) - u_d \operatorname{erf}(m) &= u_f\beta - U\beta \\
U \operatorname{erf}(m) + U\beta &= u_d \operatorname{erf}(m) + u_f\beta \\
U &= \frac{u_d \operatorname{erf}(m) + u_f\beta}{\operatorname{erf}(m) + \beta}
\end{aligned}
$$

followed by the Stefan condition:

$$
\begin{aligned}
k_2 \frac{(u_f - U)e^{-m^2}}{\mathrm{erf}(m)\sqrt{\pi\alpha_2 t}} &= \rho_2 \lambda m \sqrt{\frac{\alpha_2}{t}} \\
m e^{m^2} \mathrm{erf}(m) &= \frac{k_2(u_f - U)}{\sqrt{\pi}\rho_2\lambda\alpha_2}
\end{aligned}
$$

and from our solution for U we have:

$$
\begin{aligned}
\beta &= \frac{(U - u_d)}{(u_f - U)} \mathrm{erf}(m) \\
m e^{m^2} \beta &= \frac{(U - u_d)}{(u_f - U)} m e^{m^2} \mathrm{erf}(m) \\
&= \frac{(U - u_d)}{(u_f - U)} \frac{k_2(u_f - U)}{\sqrt{\pi}\rho_2\lambda\alpha_2} \\
&= \frac{k_2(U - u_d)}{\sqrt{\pi}\rho_2\lambda\alpha_2}
\end{aligned}
$$

adding the two gives us:

$$
\begin{aligned}
m e^{m^2} \mathrm{erf}(m) + m e^{m^2} \beta &= \frac{k_2(u_f - U)}{\sqrt{\pi}\rho_2\lambda\alpha_2} + \frac{k_2(U - u_d)}{\sqrt{\pi}\rho_2\lambda\alpha_2} \\
&= \frac{k_2(u_f - u_d)}{\sqrt{\pi}\rho_2\lambda\alpha_2} \\
m e^{m^2}(\mathrm{erf}(m) + \beta) &= \sigma \qquad \text{where} \qquad \sigma = \frac{k_2(u_f - u_d)}{\sqrt{\pi}\rho_2\lambda\alpha_2}
\end{aligned}
$$

3.3 Results

We are now in a position to calculate values for U, m, and a value for t_h, from which we can calculate the puddle length, ℓ. As usual, we can solve for m graphically (see figure 3.1) or numerically.

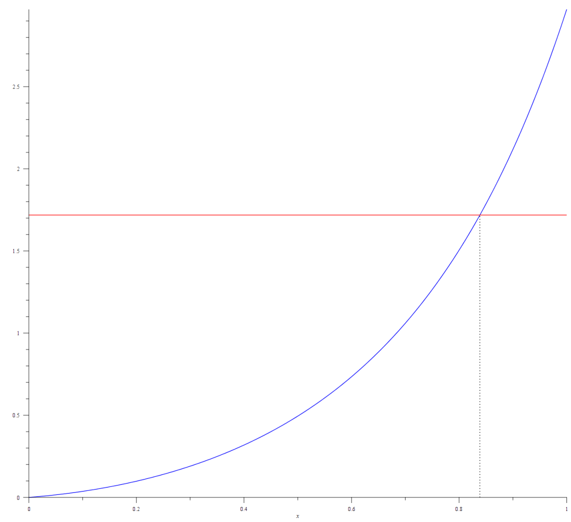

Figure 3.1: Solving for m graphically

For our numerical calculation we will use the following values:

- h 0.01 m
- V 1 m s^{-1}
- u_f 1400°C
- u_d 150°C
- λ 2.7×10^5 J kg^{-1}
- α_1 10^{-4} $m^2 s^{-1}$
- α_2 4×10^{-6} $m^2 s^{-1}$
- k_1 400 W m^{-1}°C^{-1}
- k_2 20 W m^{-1}°C^{-1}
- ρ_2 7.6×10^3 kg m^{-3}

The material of the drum is chosen to be copper because of it's good conductivity. Using Maple and the *fsolve* function we find the numerical solution for U, the teperature at the surface of the drum, to be 458.0798°C to 4 decimal places, which is much less than the melting temperature of copper, which in turn is less than the melting temperature of steel. We find the numerical solution for m to be 0.8386 to 4 decimal places. In order to calculate t_h we find the time taken for the moving boundary to reach the thickness h:

$$
\begin{aligned}
s(t_h) &= h \\
m\sqrt{4\alpha_2 t_h} &= h \\
\sqrt{4\alpha_2 t_h} &= \frac{h}{m} \\
4\alpha_2 t_h &= \frac{h^2}{m^2} \\
t_h &= \frac{h^2}{4\alpha_2 m^2}
\end{aligned}
$$

from which we find the numerical solution to t_h to be 8.8878 seconds to 4 decimal places. With a value for V set to 1 m s^{-1}, our puddle length ℓ is 8.8878 metres to 4 decimal places.

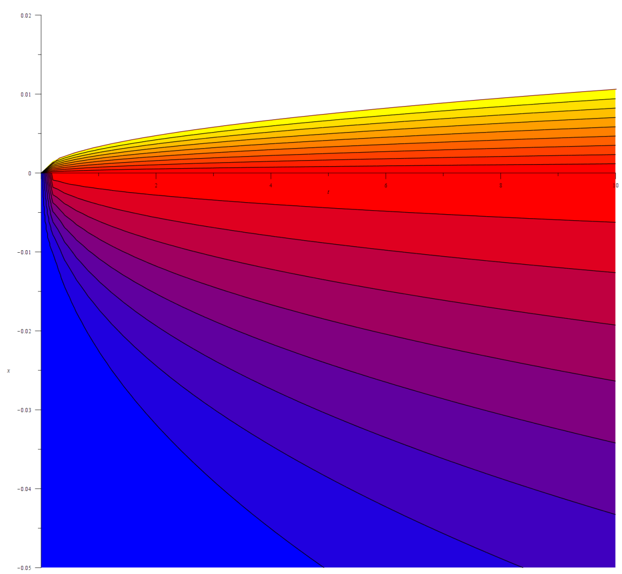

Figure 3.2: A contour plot of Temperature (u) versus x and t

In figure 3.2 we see a contour plot of the temperature for differing values of x and t, with temperature ranging from blue (colder) to red (warmer) to yellow (warmest). We see that as time advances, the solidification front advances, the solid steel cools as it approaches the drum surface, and the temperature of the drum increases, but decreases from the surface inwards.

Chapter 4

Conclusions

The calculated value for the puddle length, almost 9 metres, seems very large, especially in comparison with the thickness of the sheet, 0.01 metres, and would require a drum of an impractical size to ensure the molten steel had solidified before leaving the drum, and for this reason the approach must be deemed unfeasible. On the other hand, the approach to modelling the problem itself seems to be successful.

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