The Problem with Roast Beef

Jerry Kiely

School of Mathematical Sciences
Dublin Institute of Technology
Dublin 8

d16126734@mydit.ie

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Contents

Abstract

In this paper we will write a mathematical model, based on the heat equation, in order to verify some well known facts about roasting a piece of meat. We will derive an analytical solution, compute values from the analytic solution, compare these with values from a numerical solution, and discuss conclusions.

Chapter 1

Introduction

In this paper we will construct a mathematical model for roasting a piece of meat, assumed to be spherical, based on the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

in order to verify a number of facts, such as:

- The roasting time T is proportional to $W^{\frac{3}{2}}$, where W is the weight of the meat
- After the meat has been removed from the oven, the center temperature continues to rise around $5 - 10\,^{\circ}\mathrm{C}$

We will make a number of assumptions in order to simplify our model:

- The meat is initially at room temperature - this forms the initial condition of the solution
- The roast is spherical - in truth the roast is not spherical, but by making that assumption we can simplify the problem, and by assuming spherical symmetry we can simplify it even further
- The meat is homogenous - i.e. it has uniform (constant) density and heat parameters which are not functions of the radius or time
- There are no changes in the structure of the meat during cooking - so this is not a function of the radius or time either
- The temperature in the room and the oven are uniform - there are no variations in temperature at different points at the surface of the meat

Chapter 2

Mathematical Model

2.1 The Equation

Recall the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad 0 \leq x \leq L, \qquad t \geq 0$$

where $T(x, t)$ is the temperature at the point x and the time t, and where α is the *heat diffusivity* given by:

$$\alpha = \frac{k}{\rho . c}$$

where k is the *thermal conductivity*, ρ is the *density*, and c is the *specific heat capacity*. As we assume the meat is spherical, we transform the heat equation into spherical coordinates (see figure 2.1), and the heat equation becomes:

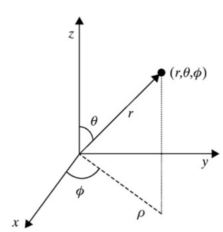

Figure 2.1: Spherical Coordinates

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 sin\theta} \frac{\partial}{\partial \theta} \left(sin\theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 sin^2\theta} \frac{\partial}{\partial \phi} \left(\frac{\partial T}{\partial \phi} \right) \right]$$

By assuming spherical symmetry, that the temperature only depends on the distance from the centre, i.e. the temperature does not depend on θ or ϕ, the equation simplifies even further:

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right), \qquad 0 \leq r \leq R, \qquad t \geq 0$$

where R is the radius of the meat.

2.2 The Boundary Conditions

In order to solve our problem we introduce the following boundary conditions:

2.2.1 Newton cooling boundary condition at the surface

Newton's Law of Cooling states that *the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of the surrounding medium* [3]. In equation form, for $r = R$:

$$-k \frac{\partial T}{\partial r}(R, t) = h(T(R, t) - T_S)$$

where k is the *thermal conductivity* mentioned earlier, h is the *heat transfer coefficient*, and T_S is the temperature of the surrounding medium (the oven or the room). So we have:

$$-k \frac{\partial T}{\partial r}(R, t)$$

is the rate at which the temperature changes on the surface and:

$$h(T(R, t) - T_S)$$

is the difference between the surface temperature and the temperature of the surrounding medium.

2.2.2 Zero-flux boundary condition at the centre

At $r = 0$ there is no heat transfer in the negative direction. In other words, the temperature has a minimum at the centre of the meat, i.e. the first derivative of the temperature with respect to r is zero:

$$\frac{\partial T}{\partial r}(0, t) = 0$$

2.3 The Initial Condition

The initial condition is that the temperature of the roast is room temperature (T_0):

$$T(r, 0) = T_0$$

2.4 Scaling

We will now scale, or non-dimensionalise, the above equations in order to allow us to concentrate on the relationship between the important parameters, and also to simplify the analysis of our model. The way we do this is to divide each physical parameter by some typical values. For example, in the case of r we divide it by the radius R of the meat:

$$\overline{r} = \frac{r}{R}, \qquad 0 \leq \overline{r} \leq 1$$

and in the case of T we divide it by 1°C:

$$\overline{T} = \frac{T}{T^*}, \qquad T^* = 1°\text{C}$$

and in the case of t we divide it by a value t^*, the value of whhich will be decided later:

$$\bar{t} = \frac{t}{t^*}$$

We now transform the PDE, BCs, and IC based on these new variables:

2.4.1 Scaling the PDE

With respect to the LHS we get:

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial(\overline{T}T^*)}{\partial(\overline{t}t^*)} \\ &= \frac{T^*}{t^*}\frac{\partial\overline{T}}{\partial\overline{t}}\end{aligned}$$

and with respect to the RHS we get:

$$\begin{aligned}\alpha\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) &= \alpha\frac{1}{\overline{r}^2R^2}\frac{\partial}{\partial(\overline{r}R)}\left(\overline{r}^2R^2\frac{\partial(\overline{T}T^*)}{\partial(\overline{r}R)}\right) \\ &= \frac{\alpha T^*}{R^2}\frac{1}{\overline{r}^2}\frac{\partial}{\partial\overline{r}}\left(\overline{r}^2\frac{\partial\overline{T}}{\partial\overline{r}}\right)\end{aligned}$$

so our equation becomes:

$$\begin{aligned}\frac{T^*}{t^*}\frac{\partial\overline{T}}{\partial\overline{t}} &= \frac{\alpha T^*}{R^2}\frac{1}{\overline{r}^2}\frac{\partial}{\partial\overline{r}}\left(\overline{r}^2\frac{\partial\overline{T}}{\partial\overline{r}}\right) \\ \frac{\partial\overline{T}}{\partial\overline{t}} &= \frac{\alpha t^*}{R^2}\frac{1}{\overline{r}^2}\frac{\partial}{\partial\overline{r}}\left(\overline{r}^2\frac{\partial\overline{T}}{\partial\overline{r}}\right) \\ &= A\frac{1}{\overline{r}^2}\frac{\partial}{\partial\overline{r}}\left(\overline{r}^2\frac{\partial\overline{T}}{\partial\overline{r}}\right)\end{aligned}$$

where A has the value:

$$A = \frac{\alpha t^*}{R^2}$$

We can now decide on a value for t^* by setting A equal to 1:

$$\begin{aligned}\frac{\alpha t^*}{R^2} &= 1 \\ \therefore \quad t^* &= \frac{R^2}{\alpha}\end{aligned}$$

which simplifies our equation considerably. To recover physical time we will need to multiply t by t^*

2.4.2 Scaling the BCs

With respect to the Newton cooling boundary condition we get:

$$\begin{aligned}
-k\frac{\partial T}{\partial r} &= h(T(R,t) - T_S) \\
-k\frac{\partial(\overline{T}T^*)}{\partial(\overline{r}R)} &= h(T^*\overline{T}(1,\overline{t}) - T_S) \\
\frac{-kT^*}{R}\frac{\partial\overline{T}}{\partial\overline{r}} &= h(T^*\overline{T}(1,\overline{t}) - T_S) \\
\frac{\partial\overline{T}}{\partial\overline{r}} &= \frac{-hR}{k}(\overline{T}(1,\overline{t}) - \frac{T_S}{T^*}) \\
\frac{\partial\overline{T}}{\partial\overline{r}} &= -Bi(\overline{T}(1,\overline{t}) - \overline{T_S})
\end{aligned}$$

where Bi is the Biot number, a parameter used frequently in heat transfer problems. It gives a measure of the ratio of heat conductivity / transfer - both at the surface and inside the body of the object being modelled.

$$Bi = \frac{hR}{k}$$

If Bi is very small, this implies that k (the thermal conductivity) is very large, and / or h (the heat transfer coefficient) is very small, which would mean the meat would heat very fast, and the temperature would be uniform throughout.

With respect to the Zero-flux boundary condition we get:

$$\begin{aligned}
\frac{\partial T}{\partial r}(0,t) &= 0 \\
\frac{\partial(\overline{T}T^*)}{\partial(\overline{r}R)}(0,t) &= 0 \\
\frac{T^*}{R}\frac{\partial\overline{T}}{\partial\overline{r}}(0,t) &= 0 \\
\therefore \frac{\partial\overline{T}}{\partial\overline{r}}(0,t) &= 0
\end{aligned}$$

2.4.3 Scaling the IC

With respect to the initial condition we get:

$$
\begin{aligned}
\overline{T}(r,0) &= \frac{T_0}{T^*} \\
&= \overline{T_0}
\end{aligned}
$$

We may now drop the bars from subsequent calculations, with the understanding that all variables are now dimensionless

2.5 Transforming the equation

We now make the transformation $u(r,t) = T(r,t) - T_S$. For our PDE we get the following for the LHS:

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= \frac{\partial (u + T_S)}{\partial t} \\
&= \frac{\partial (u)}{\partial t} + \frac{\partial (T_S)}{\partial t} \\
&= \frac{\partial u}{\partial t}
\end{aligned}
$$

and the following for the RHS:

$$
\begin{aligned}
A\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) &= A\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial (u + T_S)}{\partial r}\right) \\
&= A\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) \\
&= A\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right]
\end{aligned}
$$

combining both we get:

$$
\frac{\partial u}{\partial t} = A\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right]
$$

for the BCs and IC we get the following:

$$
\begin{aligned}
\frac{\partial u}{\partial r}(1,t) &= -Bi \cdot u(1,t) \\
\frac{\partial u}{\partial r}(0,t) &= 0 \\
u(r,0) &= T_0 - T_S
\end{aligned}
$$

2.6 Summary

Our model can now be written as follows:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= A\left[\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right], \qquad 0 \leq r \leq 1, \qquad t \geq 0 \\
\frac{\partial u}{\partial r}(1,t) &= -Bi \cdot u(1,t) \\
\frac{\partial u}{\partial r}(0,t) &= 0 \\
u(r,0) &= T_0 - T_S
\end{aligned}
$$

where our variables are:

- R the radius of the spherical piece of meat
- r the scaled (dimensionless) distance from the centre of the meat
- t the scaled (dimensionless) time
- $T(r,t)$ the scaled (dimensionless) temperature
- $u(r,t)$ $u(r,t) = T(r,t) - T_S$

and our parameters are:

- T_0 the initial temperature
- T_S the temperature of the surrounding medium
- Bi the ratio of heat transfer properties
- α the heat diffusivity of the meat
- k the thermal conductivity of the meat
- ρ the density of the meat
- c the specific heat capacity of the meat
- h the heat transfer coefficient
- m the weight of the meat

Equipped with all of the above we can now proceed with solving our problem.

Chapter 3

Analytical Solution

3.1 The solution

We are now in a position to solve this system. Let $u(r,t) = X(r)T(t)$. Substituting this into our system, it becomes:

$$
\begin{aligned}
X(r)T'(t) &= X''(r)T(t) + \frac{2}{r}X'(r)T(t) \\
\frac{T'(t)}{T(t)} &= \frac{X''(r)}{X(r)} + \frac{2}{r}\frac{X'(r)}{X(r)}
\end{aligned}
$$

As the LHS and RHS are functions of t and r respectively, both the LHS and the RHS are set equal to a seperation constant $-\lambda$:

$$
\begin{aligned}
\frac{T'(t)}{T(t)} &= -\lambda \\
\frac{X''(r)}{X(r)} + \frac{2}{r}\frac{X'(r)}{X(r)} &= -\lambda
\end{aligned}
$$

With respect to T we have:

$$
\begin{aligned}
T'(t) &= -\lambda \cdot T(t) \\
T(t) &= e^{-\lambda t}
\end{aligned}
$$

With respect to X we have:

$$\begin{aligned} X''(r) + \frac{2}{r}X'(r) &= -\lambda X(r) \\ X''(r) + \frac{2}{r}X'(r) + \lambda X(r) &= 0 \end{aligned}$$

This contains a non-constant coefficient. So our system has become:

$$\begin{aligned} X''(r) + \frac{2}{r}X'(r) + \lambda X(r) &= 0, \qquad 0 \leq r \leq 1 \\ X'(1) &= Bi \cdot X(1) \\ X'(0) &= 0 \end{aligned}$$

If we make a change of variables:

$$\begin{aligned} Y(r) &= rX(r) \\ Y'(r) &= X(r) + rX'(r) \\ Y''(r) &= 2X'(r) + rX''(r) \end{aligned}$$

This gives us the following:

$$\begin{aligned} X''(r) + \frac{2}{r}X'(r) + \lambda X(r) &= 0 \\ rX''(r) + 2X'(r) + \lambda rX(r) &= 0 \\ Y''(r) + \lambda Y(r) &= 0 \end{aligned}$$

For our boundary conditions, from the above, for $r = 1$ we have:

$$\begin{aligned} Y(1) &= X(1) \\ Y'(1) &= X(1) + X'(1) \\ \therefore X'(1) &= Y'(1) - Y(1) \end{aligned}$$

$$\begin{aligned} \therefore Y'(1) - Y(1) &= -Bi \cdot Y(1) \\ Y'(1) &= Y(1) - Bi \cdot Y(1) \\ Y'(1) &= (1 - Bi)Y(1) \end{aligned}$$

and for $r = 0$ we have:

$$\begin{aligned} Y'(0) &= X(0) + 0 \cdot X'(0) \\ &= X(0) \end{aligned}$$

but $X(0)$ is unknown so instead we use:

$$\begin{aligned} Y(0) &= 0 \cdot X(0) \\ &= 0 \end{aligned}$$

So our system becomes:

$$\begin{aligned} Y''(r) + \lambda Y(r) &= 0, \qquad 0 \leq r \leq 1 \\ Y'(1) &= (1 - Bi)Y(1) \\ Y(0) &= 0 \end{aligned}$$

which is a Sturm-Liouville problem. Solving the auxilary quadratic equation:

$$\begin{aligned} m^2 + \lambda &= 0 \\ m^2 &= -\lambda \\ m &= \pm\sqrt{-\lambda} \end{aligned}$$

if $\lambda < 0$:

$$Y(r) = Ae^{m_1 r} + Be^{m_2 r}$$

which never works in this context. If $\lambda > 0$:

$$
\begin{aligned}
m &= \pm i\sqrt{\lambda} \\
Y(r) &= A\sin(\sqrt{\lambda_1}r) + B\cos(\sqrt{\lambda_2}r)
\end{aligned}
$$

setting $r = 0$ we get:

$$
\begin{aligned}
Y(0) &= A\sin(0) + B\cos(0) \\
&= B \\
&= 0 \\
\therefore B &= 0 \\
\therefore Y(r) &= A\sin(\sqrt{\lambda}r) \\
Y'(r) &= A\sqrt{\lambda}\cos(\sqrt{\lambda}r)
\end{aligned}
$$

and setting $r = 1$ we get:

$$
\begin{aligned}
Y(1) &= A\sin(\sqrt{\lambda}) \\
Y'(1) &= A\sqrt{\lambda}\cos(\sqrt{\lambda}) \\
(1 - Bi)Y(1) &= (1 - Bi)A\sin(\sqrt{\lambda}) \\
A\sqrt{\lambda}\cos(\sqrt{\lambda}) &= (1 - Bi)A\sin(\sqrt{\lambda}) \\
\frac{\sqrt{\lambda}}{\tan(\sqrt{\lambda})} &= 1 - Bi \\
1 - \frac{\sqrt{\lambda}}{\tan(\sqrt{\lambda})} &= Bi
\end{aligned}
$$

the solutions of which are the eigenvalues of the problem. Proceeding with the solution, we recover $X(r)$:

$$
\begin{aligned}
X(r) &= \frac{Y(r)}{r} \\
&= \frac{A\sin(\sqrt{\lambda}r)}{r}
\end{aligned}
$$

We are now in a position to write $u(r,t)$ as an infinite sum of eigenfunction products:

$$\begin{aligned}
u(r,t) &= \sum_{n=1}^{\infty} X_n(r) T_n(t) \\
&= \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \frac{A \sin(\sqrt{\lambda_n} r)}{r} \\
&= \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \frac{\sin(\sqrt{\lambda_n} r)}{r}
\end{aligned}$$

Here we have absorbed A into the Fourier coefficients of the infinite sum. Now recovering $T(r,t)$:

$$\begin{aligned}
T(r,t) &= T_S + u(r,t) \\
&= T_S + \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} \frac{\sin(\sqrt{\lambda_n} r)}{r}
\end{aligned}$$

at $t = 0$:

$$\begin{aligned}
T(r,0) &= T_S + \sum_{n=1}^{\infty} c_n \frac{\sin(\sqrt{\lambda_n} r)}{r} \\
&= T_0 \\
\therefore \sum_{n=1}^{\infty} c_n \frac{\sin(\sqrt{\lambda_n} r)}{r} &= T_0 - T_S
\end{aligned}$$

In order to find the Fourier coefficients we make use of the orthogonality of the eigenfunctions:

$$
\begin{aligned}
\sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n} r) \sin(\sqrt{\lambda_m} r) &= (T_0 - T_S) \sin(\sqrt{\lambda_m} r) r \\
\int_0^1 \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n} r) \sin(\sqrt{\lambda_m} r) dr &= (T_0 - T_S) \int_0^1 \sin(\sqrt{\lambda_m} r) r dr \\
\int_0^1 c_m \sin^2(\sqrt{\lambda_m} r) dr &= (T_0 - T_S) \int_0^1 \sin(\sqrt{\lambda_m} r) r dr \\
c_m \left(\frac{2\sqrt{\lambda_m} + \sin(2\sqrt{\lambda_m})}{4\sqrt{\lambda_m}} \right) &= (T_0 - T_S) \left(\frac{\sin(\sqrt{\lambda_m}) - \sqrt{\lambda_m} \cos(\sqrt{\lambda_m})}{\lambda_m} \right) \\
c_m &= (T_0 - T_S) \frac{1}{\sqrt{\lambda_m}} \left(\frac{4(\sin(\sqrt{\lambda_m}) - \sqrt{\lambda_m} \cos(\sqrt{\lambda_m}))}{2\sqrt{\lambda_m} + \sin(2\sqrt{\lambda_m})} \right)
\end{aligned}
$$

Putting this into our equation, we get:

$$
T(r,t) = T_S + (T_0 - T_S) \sum_{m=1}^{\infty} C_m e^{-\lambda_m t} \frac{\sin(\sqrt{\lambda_m} r)}{\sqrt{\lambda_m} r}
$$

where:

$$
C_m = \frac{4(\sin(\sqrt{\lambda_m}) - \sqrt{\lambda_m} \cos(\sqrt{\lambda_m}))}{2\sqrt{\lambda_m} + \sin(2\sqrt{\lambda_m})}
$$

Note that the above equation for $T(r,t)$ is not valid for $r = 0$, but because we know that:

$$
\lim_{x \to 0} \frac{\sin(x)}{x} = 1
$$

we can rewrite it for the centre as:

$$
T_{centre}(t) = \lim_{r \to 0} T(r,t) = T_S + (T_0 - T_S) \sum_{m=1}^{\infty} C_m e^{-\lambda_m t}
$$

3.2 The values

We are now in a position to approximate values for the cooking problem by taking partial sums of the infinite series solution. First lets define some values to use:

- T_0 20°C
- T_S 180°C
- k 0.42 W/mK
- ρ 1000 kg/m^3
- c 2921 J/mK
- h 50 W/m^2K
- m 1.5 kg

We earlier showed that:

$$1 - \frac{\sqrt{\lambda}}{\tan(\sqrt{\lambda})} = Bi$$

the solutions of which are the eigenvalues of the problem. By rearranging the above, we can use Maple to find eigenvalues as the intersections of the LHS and the RHS of the below equation (see figure 3.1 for a visualisation):

$$\tan(\sqrt{\lambda}) = \frac{\sqrt{\lambda}}{1 - Bi}$$

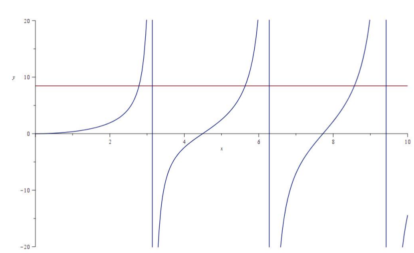

Figure 3.1: Plots of $\tan(\sqrt{\lambda})$ and $\frac{\sqrt{\lambda}}{1-Bi}$

m	λ_m	C_m
1	2.784	1.901
2	5.636	-1.667
3	8.569	1.407
4	11.568	-1.182
5	14.609	1.003

Table 3.1: Eigenvalues and corresponding coefficients

We calculate the first five eigenvalues and corresponding coefficients using maple (see table 3.1). From these we calculate the cooking time for a medium-rare roast (when the temperature at the centre reaches 60°C) to be 66 minutes approximately, and for a well-done roast (when the temperature at the centre of the roast reaches 70°C) to be 73 minutes approximately.

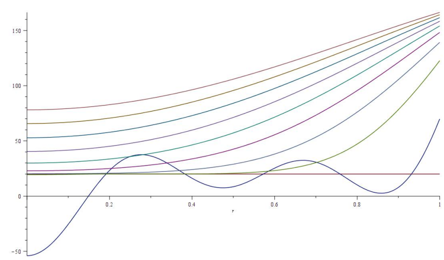

Figure 3.2: The temperature distribution at different times

See figure 3.2 for a plot of temperature distribution for different points over time in the roast. The oscilating line is for $t = 0$ and is due to inaccuracy when the infinite sum is approximated by a finite number of terms.

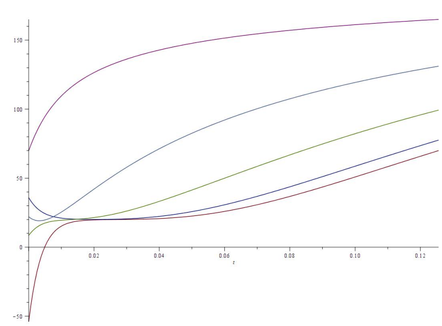

Figure 3.3: The temperature at different points in the roast

Figure 3.3 is a plot of the temperature evolution over time at different points in the roast. As can be seen, from both plots, the analytic solution doesn't seem to behave well for small t, and this is due to the fact that we are taking a finite sum of what is an an infinite series. By increasing the number of terms in the sum - by calculating more eigenvalues and corresponding coefficients - we can increase the accuracy, but $t = 0$ will always be pretty inaccurate and not behave very well.

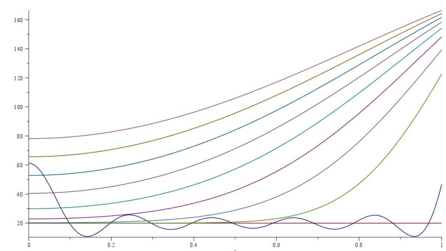

Figure 3.4: The temperature distribution at different times - using ten terms

Indeed, by doubling the number of eigenvalues and coefficients, and by increasing the sum to ten terms, our approximation does indeed improve, but the exhibitted oscilatory behaviour around $t = 0$ increases - in fact, the number of oscilations doubles (see figure 3.4).

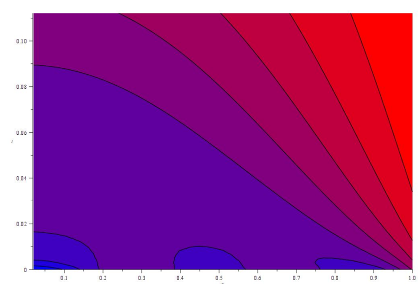

Figure 3.5: A contour plot of temperature as a function of position and time

Figure 3.5 is a contour plot of the temperature as a function of both radius and time - with the colour red representing hot regions and blue representing cold. As can be seen from the contour plot, our solution exhibits some strange behaviour - for example, the temperature from the centre to the surface appears to oscilate periodically for small t. Once again, because we are taking a finite sum as an approximation we are observing strange results.

Chapter 4

Numerical Solution

We now look at a numerical solution to the problem. We will use Maple, and specifically *pdsolve*. We give it the PDE, the BCs and the IC, and calculate values and plot the results. We will use the same values for the parameters we used in the previous section.

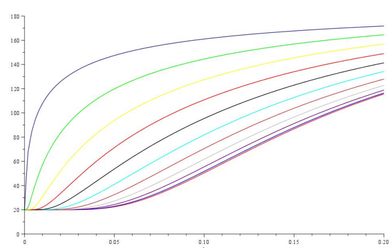

Figure 4.1: The temperature at different points in the roast during cooking

To begin with we will look at the temperature evolution at different points in the roast during cooking. In figure 4.1 we see a plot of the temperature over a two hour period with each curve representing a different point in the roast.

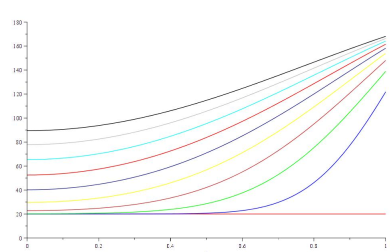

Figure 4.2: The temperature distribution at different times during cooking

Next we look at the temperature variation across the radius of the roast over the cooking period. In figure 4.2 we see a plot of the temperature at different points in the radius at different times. We can see the temperature rises sharply as you get closer to the surface.

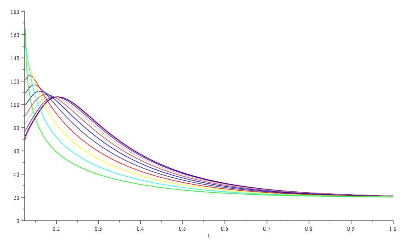

Figure 4.3: The temperature distribution at different times during cooling

Next we look at the temperature variation across the radius of the roast at different times during cooling. In figure 4.3 we see a plot of the temperature at different points in the radius over time. We can see the temperature lowers reasonably uniformly, but the temperature at, and towards, the centre continues to rise for a period after the roast has been removed from the oven.

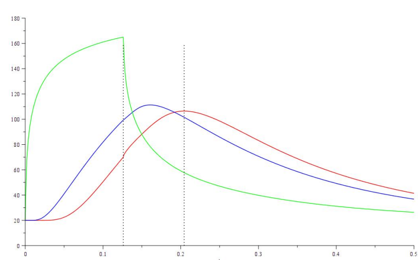

Figure 4.4: Cooking and cooling

In figure 4.4 we see a plot of the temperature at different points in the radius over time for both cooking and cooling. The two vertical lines indicate the point when the centre of the roast (in red) reaches 70°C and the point when the centre reaches it's maximum.

Note the smoothness of these plots around $r = 0$ in comparison with the plots of the analytic solution. We can clearly see that, for the cooking phase, the temperature at, and near, the surface rises quickly in comparison with the temperature at, and towards, the centre, and the temperature at the centre continues to rise to a maximum after the roast has been taken out of the oven.

Chapter 5

Further Work

5.1 Is cooking time proportional to the weight of the roast?

5.1.1 Analytic solution

Table 5.1 lists the cooking times - the times for the centre to reach 70°C - for various roast weights using data from the analytic solution:

Weight (kg)	0.5	1	1.5	2	2.5
Cooking time (min)	37.75	57.32	73.40	87.59	100.53

Table 5.1: Roast weight with coresponding cooking time

Figure 5.1 plots the points from the analytic solution with weight raised to the power of $\frac{2}{3}$.

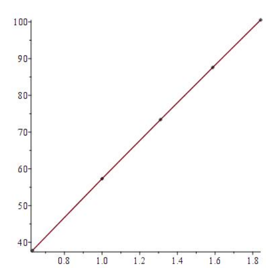

Figure 5.1: Cooking time versus weight to the power of $\frac{2}{3}$ (analytic solutuion)

5.1.2 Numeric solution

Table 5.2 lists, also for various weights, the cooking times, the time taken for the roast to reach it's maximum temperature at the centre after being taken out of the oven, and by how many degrees the temperature of the roast rises after ten minutes of cooling using the numeric solution.

Weight (kg)	Cooking time	Time to maximum	Temp. rise after 10 min
0.5	37.89	23.82	22.70
1	57.54	35.78	15.68
1.5	75.40	45.17	14.80
2	87.94	53.94	10.89
2.5	100.93	61.60	9.71

Table 5.2: Roast weight with coresponding cooking time

Figure 5.2 plots the points from the numeric solution with weight raised to the power of $\frac{2}{3}$.

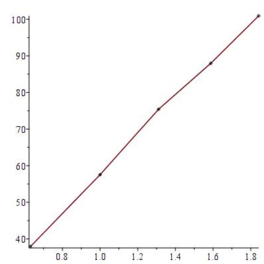

Figure 5.2: Cooking time versus weight to the power of $\frac{2}{3}$ (numeric solutuion)

5.1.3 Conclusion

As can be seen from both plots the relationship is clear, but the line going through the points from the numeric solution is less straight.

Chapter 6

Model Discussion and Conclusions

6.1 Model performance

Did our model reproduce the predictions detailed in the introduction?

- The roasting time T is proportional to $W^{\frac{3}{2}}$, where W is the weight of the meat
- After the meat has been removed from the oven, the center temperature continues to rise around $5 - 10\,^{\circ}\mathrm{C}$

For the first of our predictions we confirmed this to be the case in Chapter 5 (Further Work).

As for the second of our predictions - does the temperature at the centre continue to rise after the meat has been removed from the oven - we have shown this to be the case in Chapter 4 (Numerical Solution). Indeed our numerical solution has shown that the predicted temperature rise has been underestimated.

So our predictions have been confirmed by both models.

6.2 Comparison of solutions

With respect to our two solutions, we ask two questions:

- Were our two approaches consistent?
- Which of the two was more accurate?

By plotting both solutions against each other we can get an insight. For example, figure 6.1 plots temperature at the centre for both the analytic (red dot) and numeric (blue dashed) solution over time:

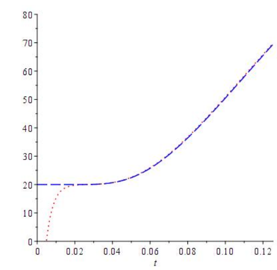

Figure 6.1: Analytic versus Numeric for $r = 0$

Clearly they deviate for t in the neighbourhood of 0. A similar plot (figure 6.2) for the surface shows more consistency:

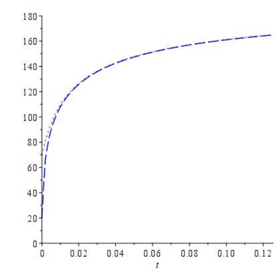

Figure 6.2: Analytic versus Numeric for $r = 1$

The real difference between the solutions can be see if we plot temperature versus radius for $t = 0$ (figure 6.3):

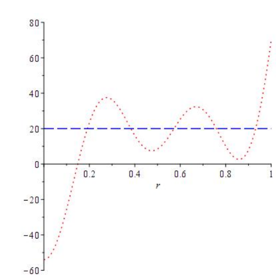

Figure 6.3: Analytic versus Numeric for $t = 0$

But this behaviour improves for $t > 0$, as can be seen if we plot temperature versus radius for $t = \frac{t_{cook}}{2}$ (figure 6.4):

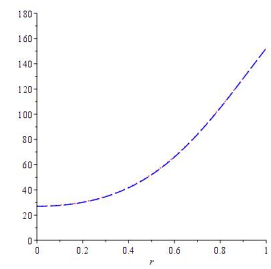

Figure 6.4: Analytic versus Numeric for $t = \frac{t_{cook}}{2}$

So certainly, with the exception of in or around the neighbourhood of $t = 0$, the solutions are very consistent. And any issues the analytic solution had around $t = 0$ were resolved by the numeric solution.

6.3 Model assumptions

In truth, many of our assumptions were oversimplifications:

- The roast is spherical - the roast is always pretty irregularly shaped, and it may in fact be more accurate to treat the roast as a cylinder
- The meat is homogenous - in fact, the meat does not have uniform (constant) density and heat parameters - which would be functions of time and radius
- There are no changes in the structure of the meat during cooking - this is certainly not the case, and these changes would also be a function of time and radius
- The temperature in the room and the oven are uniform - this may not be the case in an oven that is not fan assisted

All of the parameters used in the model would change over time, and we choose to make these assumptions in order to make our model simpler, but further research should make these parameters functions of t and r

6.4 Conclusion

Qualitatively speaking, the model performs very well. It seems to accurately predict what we expected. And quantitatively speaking, the values seem to be consistent - for example, the cooking time for a roast would agree with experience.

All in all, for what is a reasonably simple model with some reasonably naive assumptions, the model has performed surprisingly well.

Bibliography

[1] G.R. Fulford and P. Broadbridge. *Industrial Mathematics: Case Studies in the Diffusion of Heat and Matter.* Australian Mathematical Society Lecture Series. Cambridge University Press, 2002.

[2] J.D. Logan. *Applied Partial Differential Equations.* Springer Undergraduate Texts in Mathematics and Technology. Springer New York, 2004.

[3] Wikipedia. Newton's law of cooling — wikipedia, the free encyclopedia, 2017. [Online; accessed 17-March-2017].