The Parking Problem

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Contents

List of Figures

List of Tables

Listings

Abstract

The *parking problem* was proposed by Alfred Rényi in 1958 - cars, modelled as unit intervals, are placed at random upon a street of length x until no spaces that can accommodate cars remain. Rényi showed that the *jamming limit*, or expected coverage of cars on the street, is approximately 75%.

As well as being an interesting problem in it's own right, applications of the parking problem are numerous in physics and chemistry. One of the most interesting applications is in the theory of *Random Sequential Adsorption*, or RSA, which models the surface deposition of particles upon a substrate.

There are a number of approaches to this problem in one dimensions. In this paper we will look at three of the approaches - an elementary treatment, an approach that makes use of Laplace Transforms, and the kinetic approach that looks at the time evolution of the line filling process.

We will also look at a number of different generalizations of the problem, including *parking with overlap*, also know as *cooperative* RSA, and *the reversible parking problem*, where cars arrive and leave at different rates.

Chapter 1

Introduction

1.1 Problem Statement

Consider an interval $(0, x)$ upon which we place a segment of unit length at random. We continue by placing a second segment of unit length randomly upon the original interval, discarding the segment if it overlaps with the original one.

We continue in this fashion until we can no longer add unit segments without overlap. At each step the next position within the interval is chosen from a uniform distribution of the remaining locations within the interval.

We are interested in both the expected value of the number of unit segments contained within the interval $(0, x)$, denoted $M(x)$, and the expected filling density of unit segments within the interval, denoted $M(x)/x$.

1.2 Derivation of the Master Equation

We first look at the derivation of the master equation for $M(x)$ which forms the heart of the problem, and will be crop up in the rest of the paper.

We initially consider an interval $(0, x+1)$ and upon this place a unit segment $(t, t+1)$. This unit segment partitions the original interval into two smaller intervals - $(0, t)$ and $(t+1, x+1)$. The expected number of unit segments contained within the original interval is:

$$M(x+1) = M(t) + 1 + M(x-t)$$

where 1 represents the expectation of the added segment within the unit interval $(t, t+1)$ - in other words, we have one unit segment within the unit interval $(t, t+1)$. Integrating with respect to t we get:

$$\begin{aligned}
\int_0^x M(x+1)dt &= \int_0^x [M(t) + 1 + M(x-t)]dt \\
M(x+1)\int_0^x dt &= \int_0^x dt + \int_0^x [M(t) + M(x-t)]dt \\
M(x+1) \cdot x &= x + \int_0^x [M(t) + M(x-t)]dt \\
M(x+1) &= 1 + \frac{1}{x}\int_0^x [M(t) + M(x-t)]dt
\end{aligned}$$

as the distributions within each of the smaller intervals are uniform, and hence the same, we get:

$$M(x+1) = 1 + \frac{2}{x}\int_0^x M(t)dt \qquad (1.1)$$

changing variables we get:

$$M(x) = 1 + \frac{2}{x-1}\int_0^{x-1} M(t)dt \tag{1.2}$$

more completely, and because adding a unit segment to an interval of length less than 1 has no meaning, the equation for $M(x)$ can be written as follows:

$$M(x) = \begin{cases} 0, & \text{for } 0 \le x < 1 \\ 1, & \text{for } x = 1 \\ 1 + \frac{2}{x-1}\int_0^{x-1} M(t)dt, & \text{for } x > 1 \end{cases} \tag{1.3}$$

which has the form of a recurrence.

1.3 Applications

The main application of the parking problem is to the theory of random sequential adsorption (RSA), which models the adsorption of particles onto a solid substrate. It is important to make the distinction between *adsorption* and *absorption* - adsorption involves the surface of the material involved, whereas absorption involves the full volume of the material.

Some examples of RSA are adsorption of gas molecules, polymer chains, latex particles, bacteria, proteins, and colloidal particles (insoluble particles contained within a suspension) onto a surface. Another important application is when applied to genome sequencing where a newly-arriving sequenced clone is allowed to partially overlap an existing sequenced clone - in which case the parking problem with overlap the most relevant model. In some cases adsorbed particles can also be released from a surface, a process know as *desorption*, and in this case the reversible parking problem is the most relevant model.

In order to maximize the adsorption of particles, a clear understanding of the problem is required, and to that end we need to be able to faithfully model the process. This paper will offer a survey of the approaches to modelling the one-dimensional parking problem, and provide a different approach to both validate the approaches used to model the process, and to calculate the associated constants.

Chapter 2

Weiner's Approach

2.1 Finding the Jamming Limit

We first look at Weiner's elementary approach to finding the jamming limit C_R (see [5]). Using our previously derived master equation (1.3) Weiner solved for $M(x)$ recursively, giving us the following results:

$$M(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x \in [1,2) \\ 3 - \dfrac{2}{x-1} & \text{if } x \in [2,3) \\ 7 - \dfrac{10 - 4\ln(x-2)}{x-1} & \text{if } x \in [3,4) \end{cases} \tag{2.1}$$

and so on - the calculations becoming increasingly difficult, and frankly tedious, after this point. In table 2.1 we have calculated $M(x)/x$ for values of x close to

x	$M(x)/x$
0.99	0
1.99	0.5025125628
2.99	0.6672156769
3.99	0.6854478541
4.99	0.6969110964
5.99	0.7054612992
6.99	0.7114892709

Table 2.1: $M(x)/x$

the upper limit of each interval. As can be seen, the values of $M(x)/x$ seem to be approaching a limit. This can be seen more clearly in figure 2.1.

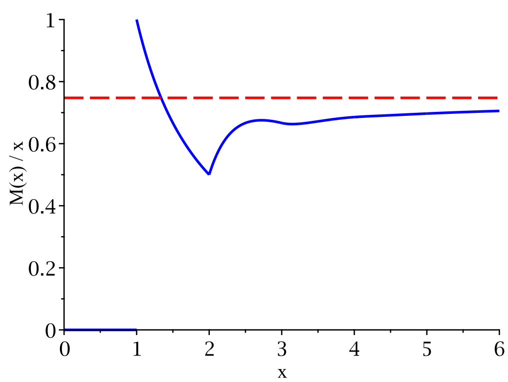

Figure 2.1: Rényi's parking constant

2.2 Existence of the Jamming Limit

In attempting to show that this jamming limit exists, Weiner took a flawed approach. In order to understand the flaw we trace his steps and return to equation 1.1. Dividing across by x we get:

$$\frac{M(x+1)}{x} = \frac{1}{x} + \frac{2}{x^2}\int_0^x M(t)dt \tag{2.2}$$

differentiating with respect to x:

$$\left(\frac{M(x+1)}{x}\right)' = -\frac{1}{x^2} - \frac{4}{x^3}\int_0^x M(t)dt + \frac{2}{x^2}M(x)$$

looking at the integral, and making use of the fact that $0 \leq M(x) \leq x$ we find that:

$$\begin{aligned}\int_0^x M(t)dt &\leq \int_0^x tdt \\ &\leq \frac{x^2}{2}\end{aligned}$$

making use of both we find:

$$\begin{aligned}\left(\frac{M(x+1)}{x}\right)' &= -\frac{1}{x^2} - \frac{4}{x^3}\int_0^x M(t)dt + \frac{2}{x^2}M(x) \\ &\leq -\frac{1}{x^2} - \frac{4}{x^3}\cdot\frac{x^2}{2} + \frac{2}{x^2}\cdot x \\ &\leq -\frac{1}{x^2} - \frac{2}{x} + \frac{2}{x} \\ &\leq -\frac{1}{x^2}\end{aligned}$$

$$= \mathcal{O}\left(\frac{1}{x^2}\right)$$

and hence:

$$\lim_{x\to\infty}\left(\frac{M(x+1)}{x}\right)' = 0$$

which according to Weiner implies that:

$$\lim_{x\to\infty}\frac{M(x)}{x} = C_R$$

this, however, is incorrect, as a simple counterexample will demonstrate. We consider:

$$\begin{aligned}
f(x) &= \sin(\ln(x)) \\
f'(x) &= \frac{\cos(\ln(x))}{x} \\
\lim_{x\to\infty} f'(x) &= \lim_{x\to\infty}\frac{\cos(\ln(x))}{x} \\
&= 0
\end{aligned}$$

but clearly:

$$
\begin{aligned}
\lim_{x\to\infty} \sin(\ln(x)) &= [-1, 1] \\
&\neq \text{a constant}
\end{aligned}
$$

so this is an unsatisfactory justification for the existence of the limit. Of course if his approach had been to show that:

$$\left(\lim_{x\to\infty} \frac{M(x+1)}{x}\right)' = 0$$

his conclusions would have been correct. However, the limit of the derivative of a function is not always the same as the derivative of the limit of a function. As both derivatives (and integrals) are themselves limits, the issue here is if the respective limits are independent. As both the limit, and the integration, are over x, then this presents a problem.

2.3 Bounds for the Jamming Limit

Weiner did not calculate the limit, but instead provided a method for finding increasingly narrower bounds for the limit. Let us assume for now that:

$$\lim_{x\to\infty} \frac{M(x)}{x} = C_R$$

in order to find bounds for C_R Weiner sought to sandwich $M(x)$ between two linear functions:

$$L_{a_1}(x) \leq M(x) \leq L_{a_2}(x) \tag{2.3}$$

and find the limits of each of these linear functions as $x \to \infty$. Weiner found an initial lower bound by finding, for $x \geq 1$, the linear function of the form $L(x) = ax + b$ that satisfied the following equation:

$$\begin{aligned}
L(x+1) &= 1 + \frac{2}{x}\int_1^x L(t)dt \\
ax + (a+b) &= 1 + \frac{2}{x}\int_1^x (at+b)dt \\
ax + (a+b) &= 1 + \frac{2}{x}\left[\frac{at^2}{2} + bt\right]_{t=1}^{x} \\
ax + (a+b) &= 1 + ax + 2b - \frac{a+2b}{x} \\
(a+b) &= 1 + 2b - \frac{a+2b}{x} \\
(a+b)x &= x + 2bx - (a+2b) \\
(b-a+1)x &= a + 2b
\end{aligned}$$

noting that both sides of the equation must equal 0, we arrive at two simultaneous equations:

$$a - b = 1$$

$$a + 2b \quad = \quad 0$$

from which we find the coefficients of our required linear function:

$$L(x) = \frac{2}{3}x - \frac{1}{3}$$

We know that:

$$L(x) \leq M(x) = 1 \quad \text{for } 1 \leq x < 2$$

as $M(x)$ is increasing, we see that, for $x \geq 1$:

$$\frac{L(x)}{x} \leq \frac{M(x)}{x}$$

and hence:

$$\lim_{x \to \infty} \frac{L(x)}{x} \leq \lim_{x \to \infty} \frac{M(x)}{x}$$

which gives us:

$$\frac{2}{3} \leq C_R$$

Weiner went on to find two linear functions that sandwiched $M(x)$ with a little more accuracy. The two functions found were:

$$\begin{aligned} L_{a_1}(x) &\equiv 0.7432x - 0.2568 \\ L_{a_2}(x) &\equiv 0.75x - 0.25 \end{aligned}$$

with the property found in equation 2.3 :

$$0.7432x - 0.2568 \leq M(x) \leq 0.75x - 0.25$$

dividing across by x and taking the limit as $x \to \infty$:

$$0.7432 \leq C_R \leq 0.75$$

which gives a reasonably good indication of the bounds of C_R.

2.4 Remarks

Weiner made use of the following theorem and lemma (both stated without proof):

Theorem. *Define $L_a(x) \equiv ax + a - 1$. If for some $t > 0$, $L_a(x) \leq M(x)$ ($L_a(x) \geq M(x)$) for $t \leq x \leq t + 1$, then $L_a(x) \leq M(x)$ ($L_a(x) \geq M(x)$) for all $x \geq t$*

Lemma. $M'(x) > 0$ *for* $x \geq 2$

to justify his result.

While Weiner's approach is interesting from the point of view of it's elementary nature, it is not a particularly satisfying approach due to the error made in his justification of the existence of the limit, and also because of the "trial-and-error method" used to find the bounds.

But while his assumption about the existence of the limit is flawed, he does provide an interesting, albeit dated, approach to finding the limit. His work has little more than novelty value, and is included in the survey because of it's historical interest - to see how the understanding of the problem has progressed.

Chapter 3

Rényi's Approach

3.1 Solution of the Delay Differential Equation

A more satisfying approach is Rényi's solution to the delay differential equation derived from the master equation (see [6]). We start with the familiar form:

$$M(x+1) = 1 + \frac{2}{x}\int_0^x M(t)dt \tag{3.1}$$

then multiply both sides by x:

$$xM(x+1) = x + 2\int_0^x M(t)dt$$

and differentiate the result with respect to x:

$$xM'(x+1) + M(x+1) = 1 + 2M(x) \quad \text{for} \quad x > 0 \tag{3.2}$$

We now consider the following:

$$\varphi(s) = \int_0^\infty M(x) e^{-sx} dx \tag{3.3}$$

which is the Laplace transform of $M(x)$. We will show that $\varphi(s)$ takes the form:

$$\varphi(s) = \frac{e^{-s}}{s^2} \int_s^\infty \exp\left(-2\int_s^t \frac{1-e^{-u}}{u} du\right) dt \tag{3.4}$$

Returning to equation 3.2 with the initial condition:

$$M(x) = 0 \quad \text{for} \quad 0 \le x < 1$$

we multiply across by e^{-sx} and integrating:

$$\begin{aligned} \int_0^\infty xM'(x+1)e^{-sx}dx + \int_0^\infty M(x+1)e^{-sx}dx &= \int_0^\infty e^{-sx}dx + \int_0^\infty 2M(x)e^{-sx}dx \\ &= -\frac{e^{-sx}}{s}\bigg|_{x=0}^\infty + 2\int_0^\infty M(x)e^{-sx}dx \\ &= \frac{1}{s} + 2\varphi(s) \end{aligned}$$

We take each part of the left hand side separately:

$$\begin{aligned}\int_0^\infty M(x+1)e^{-sx}dx &= \int_1^\infty M(t)e^{-s(t-1)}dt \\ &= e^s \int_1^\infty M(t)e^{-st}dt \\ &= e^s \varphi(s)\end{aligned}$$

making use of our initial condition. Looking at the other integral, and integrating by parts:

$$\begin{aligned}\int_0^\infty xM'(x+1)e^{-sx}dx &= -\frac{d}{ds}\left(\int_0^\infty M'(x+1)e^{-sx}dx\right) \\ &= -\frac{d}{ds}\left(M(x+1)e^{-sx}\Big|_{x=0}^{\infty} + s\int_0^\infty M(x+1)e^{-sx}dx\right) \\ &= -\frac{d}{ds}\left(-M(1) + se^s\varphi(s)\right) \\ &= -\frac{d}{ds}\left(se^s\varphi(s)\right)\end{aligned}$$

so our delay differential equation becomes:

$$\begin{aligned}-\frac{d}{ds}\left(se^s\varphi(s)\right) + e^s\varphi(s) &= \frac{1}{s} + 2\varphi(s) \\ -\frac{d}{ds}\left(se^s\varphi(s)\right) &= \frac{1}{s} + 2\varphi(s) - e^s\varphi(s)\end{aligned}$$

$$\frac{d}{ds}\left(se^s\varphi(s)\right) = \varphi(s)(e^s-2)-\frac{1}{s}$$

we now make the substitution $w(s)=e^s\varphi(s)$. Our equation becomes:

$$\begin{aligned}\frac{d}{ds}\left(sw(s)\right) &= w(s)(1-2e^{-s})-\frac{1}{s}\\ w(s)+sw'(s) &= w(s)-2w(s)e^{-s}-\frac{1}{s}\\ sw'(s) &= -2w(s)e^{-s}-\frac{1}{s}\end{aligned}$$

solving this inhomogeneous equation gives us:

$$w(s)=\frac{1}{s^2}\int_s^\infty \exp\left(-2\int_s^t \frac{1-e^{-u}}{u}du\right)dt$$

which is another form of equation 3.4.

3.2 The Computation of the Limit

Rényi provides the following theorem:

Theorem. *One has the limit*

$$\lim_{x\to\infty}\frac{M(x)}{x}=C_R$$

with

$$C_R=\int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}du\right)dt$$

In order to compute the limit, Rényi makes use of the following *Tauberian theorem* (stated without proof):

Theorem. *if f is positive and integrable over every finite interval $(0,T)$ and $e^{-st}f(t)$ is integrable over $(0,\infty)$ for any $s>0$ and if*

$$g(s)=\int_0^\infty e^{-st}f(t)dt \simeq Hs^{-\beta} \text{ as } s\to 0$$

where $\beta>0$ and $H>0$ then when $x\to\infty$, we have

$$F(x)=\int_0^x f(t)dt \simeq \frac{H}{\Gamma(1+\beta)}x^\beta \text{ as } x\to\infty$$

Proof. If we re-arrange equation 3.4 we have:

$$s^2\varphi(s)=e^{-s}\int_s^\infty \exp\left(-2\int_s^t \frac{1-e^{-u}}{u}du\right)dt$$

taking the limit as $s\to 0$:

$$\begin{aligned}\lim_{s\to 0} s^2\varphi(s) &= \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u}du\right)dt \\ &= C_R\end{aligned}$$

from the definition of $\varphi(s)$ (equation 3.3):

$$\int_0^\infty M(x)e^{-sx}dx = \frac{C_R}{s^2}$$

as $s \to 0$. Applying the above theorem, we get:

$$\begin{aligned}
\int_0^x M(t)dt &= \frac{C_R x^2}{\Gamma(3)} \\
&= \frac{C_R x^2}{2} \\
\frac{2}{x^2}\int_0^x M(t)dt &= C_R
\end{aligned}$$

as $x \to \infty$. Returning to the familiar equation 3.1 and dividing across by x, and taking the limit as $x \to \infty$, we have:

$$\begin{aligned}
\lim_{x\to\infty} \frac{M(x)}{x} &= \lim_{x\to\infty} \left(\frac{1}{x} + \frac{2}{x(x-1)} \int_0^{x-1} M(t)dt \right) \\
&= C_R
\end{aligned}$$

which is the required limit. □

3.3 Remarks

While certainly more interesting than Weiner's approach - indeed an expression for the jamming limit is derived - this author is still left feeling dissatisfied by the approach taken. The Laplace transform of the master equation is introduced without any of it's specific features being used - for example, the reduction of an analytic problem to an algebraic problem, and no inverse step is performed - it is merely used as a convenient form in order to apply a *Tauberian theorem*, and hence the reference to the Laplace Transform could have been omitted.

Chapter 4

Kinetic Approach

4.1 Derivation of the Rate Equations

We now treat the problem from the point of view of it's time evolution, and look at the results as $t \to \infty$ (see [1]). We assume a line of fixed length L with $L \to \infty$. We are interested in the distribution of gaps, the expected total gap length, and consequently the car coverage. Let X_t be a random variable that represents the length of a gap at time t, with $N(x,t)$ the number of gaps less than or equal to x, and $N(t)$ the total number of gaps. We define the gap length distribution $F(x,t)$ as follows:

$$P(X_t \leq x) = F(x,t) = \frac{N(x,t)}{N(t)}$$

here $P(A)$ represents the probability of event A occurring. The gap length density function $f(x,t)$ is then:

$$f(x,t) = \frac{\partial F}{\partial x} = \frac{1}{N(t)} \frac{\partial N}{\partial x}(x,t) \tag{4.1}$$

therefore the probability that a gap has length between a and b is:

$$\int_a^b f(x,t)dx = \frac{1}{N(t)} \int_a^b \frac{\partial N}{\partial x}(x,t)dx$$

which is simply the number of gaps with length between a and b divided by the total number of gaps at time t. We define the coverage function, $\theta(t)$, the proportion of L occupied by cars as:

$$\theta(t) = \frac{N(t)}{L}$$

i.e. the total number of gaps, which is also the total number of cars of unit length, divided by the line length. We define the expectation of X_t, the average length of a gap, as:

$$E(X_t) = \int_0^\infty xf(x,t)dx$$

We now define the gap density function, $P(x,t)$, as:

$$P(x,t) = \frac{1}{L}\frac{\partial N}{\partial x}(x,t) \tag{4.2}$$

where L is the length of the line. Therefore the proportion of gaps with length between a and b is:

$$\int_a^b P(x,t)dx = \frac{1}{L} \int_a^b \frac{\partial N}{\partial x}(x,t)dx$$

combining equations 4.1 and 4.2 above we have:

$$f(x,t) = \frac{L \cdot P(x,t)}{N(t)}$$

from which we get an expression, in terms of $P(x,t)$, for the expected total gap length at time t:

$$\begin{aligned} N(t) \cdot E(X_t) &= N(t) \int_0^\infty x f(x,t) dx \\ &= L \int_0^\infty x P(x,t) dx \end{aligned}$$

which is the expected total gap length at time t. Returning to the coverage function $\theta(t)$, we now express it in terms of the above expected total gap length by observing that:

$$\theta(t) = 1 - \frac{N(t) \cdot E(X_t)}{L}$$

from which we get:

$$\theta(t) = 1 - \int_0^\infty x P(x,t) dx$$

and noting that the sum pf all gap lengths and car lengths gives us the total line length L, we find a new expression for $\theta(t)$:

$$
\begin{aligned}
L &= L\int_0^\infty (x+1)P(x,t)dx \\
1 &= \int_0^\infty (x+1)P(x,t)dx \\
&= \int_0^\infty xP(x,t)dx + \int_0^\infty P(x,t)dx \\
1 - \int_0^\infty xP(x,t)dx &= \int_0^\infty P(x,t)dx
\end{aligned}
$$

from which we see that:

$$
\theta(t) = \int_0^\infty P(x,t)dx
$$

Returning to our gap density function, $P(x,t)$, we form a rate equation using our expressions for the gap density function:

$$
\frac{\partial P(x,t)}{\partial t} = \begin{cases} 2\int_{x+1}^\infty P(y,t)dy & \text{for } x < 1 \\ -(x-1)P(x,t) + 2\int_{x+1}^\infty P(y,t)dy & \text{for } x \geq 1 \end{cases}
$$

which is composed of creation and destruction terms. The creation term may be be understood by considering a gap of length $y \geq x+1$, there are exactly two ways to park within this gap that will result in a gap of length x. The destruction term, which only appears in the case where $x \geq 1$, deals with the case where an existing gap of length x is destroyed by parking a car within it

- in this case the remaining length becomes $x-1$, and this destruction can be accomplished in $P(x,t)$ ways.

4.2 Solving the Rate Equations

We will now solve the rate equation using the following two initial conditions:

$$P(x,0) = 0 \quad \forall x$$

and:

$$\lim_{t \to 0} \int_0^\infty x P(x,t) dx = 1$$

We make the following substitution:

$$P(x,t) = A(t) e^{-(x-1)t}$$

and begin by looking at our initial conditions. The first becomes:

$$\begin{aligned} P(x,0) &= A(0)e^0 \\ &= A(0) \\ &= 0 \end{aligned}$$

and the second becomes:

$$\begin{aligned}\lim_{t\to 0}\int_0^\infty xP(x,t)dx &= \lim_{t\to 0}\int_0^\infty xA(t)e^{-(x-1)t}dx \\ &= \lim_{t\to 0} A(t)\int_0^\infty xe^{-(x-1)t}dx\end{aligned}$$

integrating by parts we get:

$$\begin{aligned}\int_0^\infty xe^{-(x-1)t}dx &= \left.\frac{xe^{-(x-1)t}}{-t}\right|_{x=0}^{\infty} - \int_0^\infty \frac{e^{-(x-1)t}}{-t}dx \\ &= \left.\frac{e^{-(x-1)t}}{-t^2}\right|_{x=0}^{\infty} \\ &= \frac{e^t}{t^2}\end{aligned}$$

and putting the result into the above gives us:

$$\begin{aligned}\lim_{t\to 0} A(t)\int_0^\infty xe^{-(x-1)t}dx &= 1 \\ \lim_{t\to 0}\frac{A(t)e^t}{t^2} &= 1 \\ \therefore \quad \lim_{t\to 0}\frac{A(t)}{t^2} &= 1\end{aligned}$$

For $x \geq 1$:

$$\frac{\partial}{\partial t}(A(t)e^{-(x-1)t}) = -(x-1)A(t)e^{-(x-1)t} + 2\int_{x+1}^\infty A(t)e^{-(y-1)t}dy$$

$$
\begin{aligned}
A'(t)e^{-(x-1)t} - (x-1)A(t)e^{-(x-1)t} &= -(x-1)A(t)e^{-(x-1)t} + 2A(t) \left.\frac{e^{-(y-1)t}}{-t}\right|_{y=x+1}^{\infty} \\
A'(t)e^{-(x-1)t} &= 2A(t)\frac{e^{-t}}{t}e^{-(x-1)t} \\
A'(t) &= 2A(t)\frac{e^{-t}}{t}
\end{aligned}
$$

which is an ODE. If we make a further substitution $A(t) = t^2F(t)$ our initial condition becomes:

$$
\begin{aligned}
\lim_{t\to 0}\frac{A(t)}{t^2} &= 1 \\
\lim_{t\to 0}\frac{t^2F(t)}{t^2} &= 1 \\
\lim_{t\to 0}F(t) &= 1 \\
F(0) &= 1
\end{aligned}
$$

and our equation becomes:

$$
\begin{aligned}
A'(t) &= 2A(t)\frac{e^{-t}}{t} \\
2tF(t) + t^2F'(t) &= 2tF(t)e^{-t}
\end{aligned}
$$

$$\begin{aligned}
2F(t) + tF'(t) &= 2F(t)e^{-t} \\
tF'(t) &= 2F(t)e^{-t} - 2F(t) \\
&= F(t)\left(2(e^{-t} - 1)\right) \\
F'(t) &= F(t)\left(2\frac{(e^{-t} - 1)}{t}\right) \\
F'(t) + F(t)\left(2\frac{(1 - e^{-t})}{t}\right) &= 0
\end{aligned}$$

we solve this using the integrating factor:

$$I(t) = \exp\left(2\int_0^t \frac{(1 - e^{-\tau})}{\tau} d\tau\right)$$

which gives us:

$$\begin{aligned}
F'(t) \cdot I(t) + F(t) \cdot I(t)\left(2\frac{(1 - e^{-t})}{t}\right) &= 0 \\
F'(t) \cdot I(t) + F(t) \cdot I'(t) &= 0 \\
\frac{d}{dt}(F(t) \cdot I(t)) &= 0 \\
F(t) \cdot I(t) &= C \\
F(t) &= \frac{C}{I(t)}
\end{aligned}$$

we make use of our initial condition $F(0) = 1$ and the fact that $I(0) = 1$ to find C:

$$\begin{aligned} F(0) &= \frac{C}{I(0)} \\ C &= 1 \end{aligned}$$

and hence:

$$F(t) = \exp\left(-2\int_0^t \frac{(1-e^{-\tau})}{\tau} d\tau\right)$$

And for $x < 1$:

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} &= 2\int_{x+1}^{\infty} P(y,t)dy \\ &= 2\int_{x+1}^{\infty} t^2 F(t) e^{-(y-1)t} dy \\ &= 2t^2 F(t) \int_{x+1}^{\infty} e^{-(y-1)t} dy \\ &= 2t^2 F(t) \left. \frac{e^{-(y-1)t}}{-t} \right|_{y=x+1}^{\infty} \\ &= 2tF(t) \left. e^{-(y-1)t} \right|_{y=x+1}^{\infty} \end{aligned}$$

$$= \quad 2tF(t)e^{-xt}$$

$$\therefore \quad P(x,t) \quad = \quad 2\int_0^t \tau F(\tau)e^{-x\tau}d\tau$$

So our gap density function can be expressed as follows:

$$P(x,t) = \begin{cases} 2\int_0^t \tau F(\tau)e^{-x\tau}d\tau & \text{for } x < 1 \\ \\ t^2F(t)e^{-(x-1)t} & \text{for } x \geq 1 \end{cases}$$

with $F(t)$ as above. Returning to our coverage function, and making use of the above expression for the gap density function, we have:

$$\theta(t) = \int_0^1 \left(2\int_0^t \tau F(\tau)e^{-x\tau}d\tau\right)dx + \int_1^\infty t^2F(t)e^{-(x-1)t}dx$$

Differentiating with respect to t we get:

$$\frac{d\theta}{dt} = \frac{\partial}{\partial t}\int_0^1 \left(2\int_0^t \tau F(\tau)e^{-x\tau}d\tau\right)dx + \frac{\partial}{\partial t}\int_1^\infty t^2F(t)e^{-(x-1)t}dx$$

We will deal with each part of the right hand side separately. Starting with the first part:

$$\frac{\partial}{\partial t}\int_0^1 \left(2\int_0^t \tau F(\tau)e^{-x\tau}d\tau\right)dx \quad = \quad \int_0^1 \left(2\frac{\partial}{\partial t}\int_0^t \tau F(\tau)e^{-x\tau}d\tau\right)dx$$

$$
\begin{aligned}
&= \int_0^1 \left(2tF(t)e^{-xt}\right) dx \\
&= 2tF(t) \int_0^1 e^{-xt} dx \\
&= 2tF(t) \left. \frac{e^{-xt}}{-t} \right|_{x=0}^{1} \\
&= 2F(t)(1 - e^{-t})
\end{aligned}
$$

moving on to the second part:

$$
\begin{aligned}
\frac{\partial}{\partial t} \int_1^\infty t^2 F(t) e^{-(x-1)t} dx &= \int_1^\infty \frac{\partial}{\partial t} \left(t^2 F(t) e^{-(x-1)t}\right) dx \\
&= \int_1^\infty \left(2tF(t)e^{-(x-1)t} + t^2 F'(t) e^{-(x-1)t} + t^2 F(t) - (x-1) e^{-(x-1)t}\right) dx \\
&= \int_1^\infty 2tF(t) e^{-(x-1)t} dx \\
&\quad + \int_1^\infty t^2 F'(t) e^{-(x-1)t} dx \\
&\quad + \int_1^\infty t^2 F(t) - (x-1) e^{-(x-1)t} dx
\end{aligned}
$$

taking each part of this integral separately:

$$
\int_1^\infty 2tF(t) e^{-(x-1)t} dx = 2tF(t) \int_1^\infty e^{-(x-1)t} dx
$$

$$
\begin{aligned}
&= 2tF(t) \left. \frac{e^{-(x-1)t}}{-t} \right|_{x=1}^{\infty} \\
&= 2F(t)
\end{aligned}
$$

followed by:

$$
\begin{aligned}
\int_1^{\infty} t^2 F'(t) e^{-(x-1)t} dx &= t^2 F'(t) \int_1^{\infty} e^{-(x-1)t} dx \\
&= t^2 F'(t) \left. \frac{e^{-(x-1)t}}{-t} \right|_{x=1}^{\infty} \\
&= tF'(t) \\
&= -2F(t)(1 - e^{-t})
\end{aligned}
$$

here we have made use of the fact that:

$$
F'(t) = -2F(t) \frac{(1 - e^{-t})}{t}
$$

and:

$$
\begin{aligned}
\int_1^{\infty} t^2 F(t) - (x-1) e^{-(x-1)t} dx &= t^2 F(t) \int_1^{\infty} -(x-1) e^{-(x-1)t} dx \\
&= t^2 F(t) \left. \frac{xte^{-(x-1)t} - te^{-(x-1)t} + e^{-(x-1)t} - 1}{t^2} \right|_{x=1}^{\infty}
\end{aligned}
$$

$$= -F(t)$$

finally putting it all together we get:

$$\frac{d\theta}{dt} = 2F(t)(1-e^{-t}) + 2F(t) - 2F(t)(1-e^{-t}) - F(t)$$

$$= F(t)$$

$$\therefore \quad \theta(t) = \int_0^t F(\tau)d\tau$$

We have derived an expression for the coverage function solely in terms of t, and in a very satisfying way. In figure 4.1 we have used the above expression to plot the coverage function. As expected, $\theta(t)$ tends towards C_R, here denoted with a dashed red line. C_R was calculated by evaluating $\theta(t)$ as $t \to \infty$.

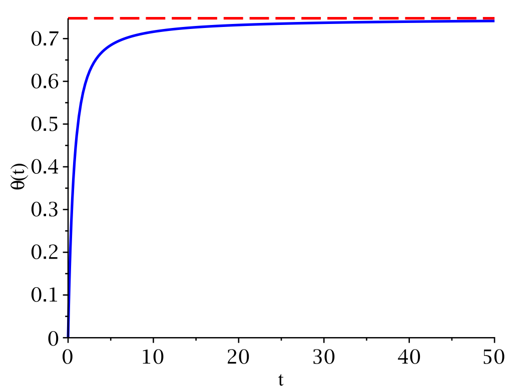

Figure 4.1: The coverage function

We now look at the behaviour of the gap density function, $P(x,t)$, as a function of x for different values of t.

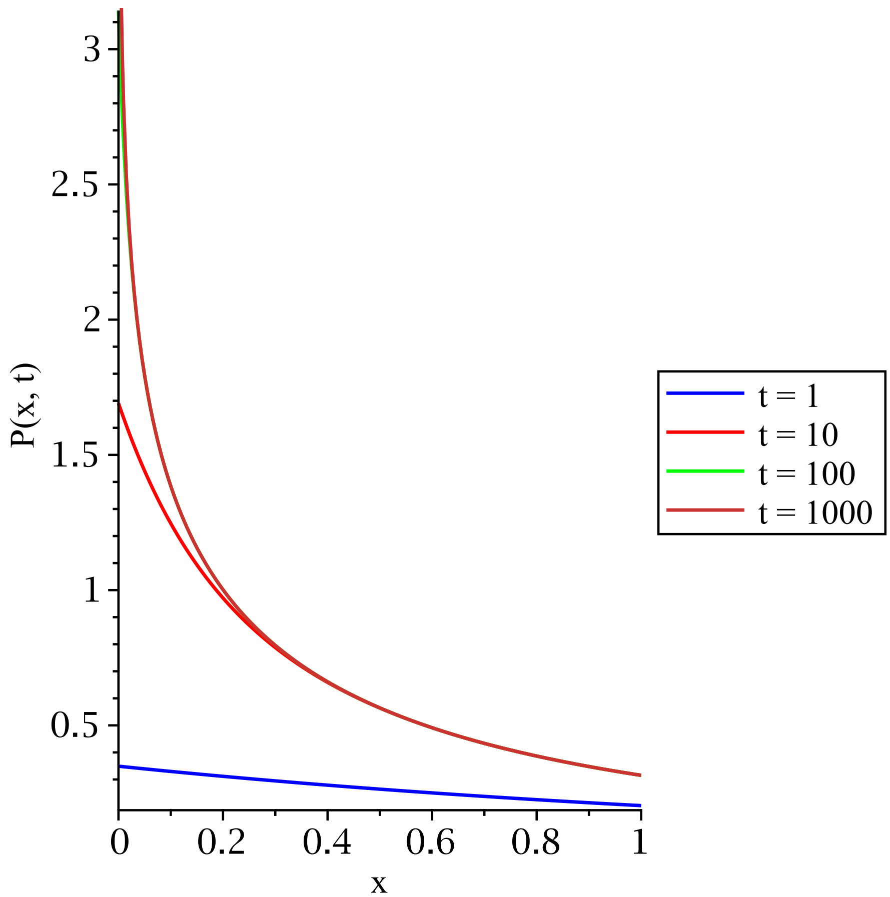

Figure 4.2: The gap density function for $x < 1$

In figure 4.2 we see the behaviour of the gap density function for $x < 1$ and for a range of values of t. As can be seen, as t increases, the density of smaller gaps increases sharply, as is to be expected due to the increasing fragmentation, but the density of larger gaps within the range becomes more constant, again to be expected because these larger gaps are still less than the size of a car, so cannot be occupied, and hence remain unchanged.

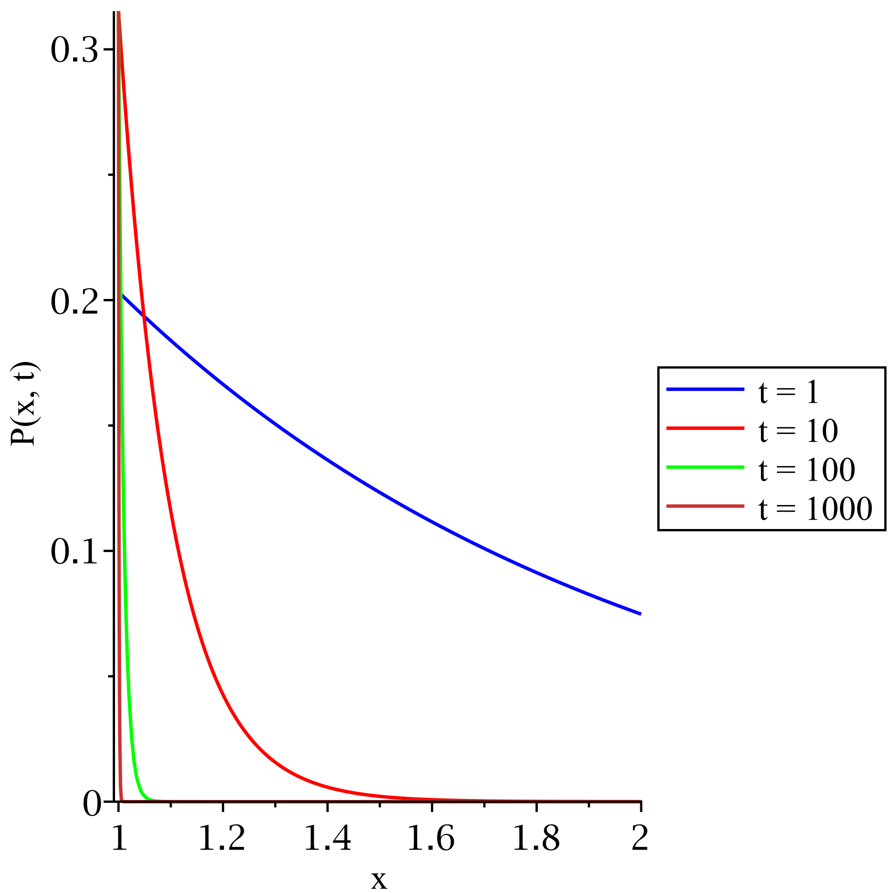

Figure 4.3: The gap density function for $x \geq 1$

In figure 4.3 we see the behaviour of the gap density function for $x \geq 1$ and for a range of values of t. As can be seen, as t increases, the density of smaller gaps increases sharply, again as is to be expected due to the increasing fragmentation, but the density of larger gaps within the range reduces, again to be expected, because these larger gaps are destroyed by cars, and all that remain are gaps not large enough to accommodate cars.

We next look at the behaviour of the gap density function, $P(x, t)$, as a function of t for different values of x.

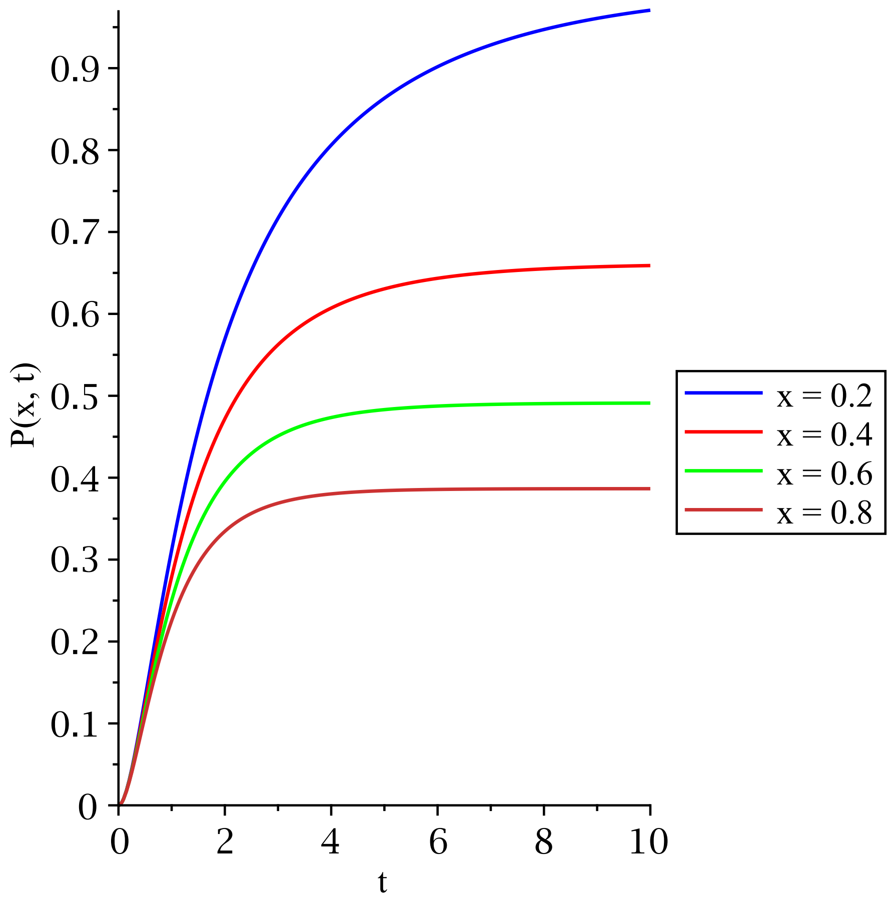

Figure 4.4: The gap density function

In figure 4.4 we see the behaviour of the gap density function for a range of values of $x < 1$. As can be seen, as t increases, the density of these smaller gaps increases to a limit. This is to be expected due to the fact that as the parking process continues, eventually gaps, which are less than the size of a car, remain and can never be destroyed.

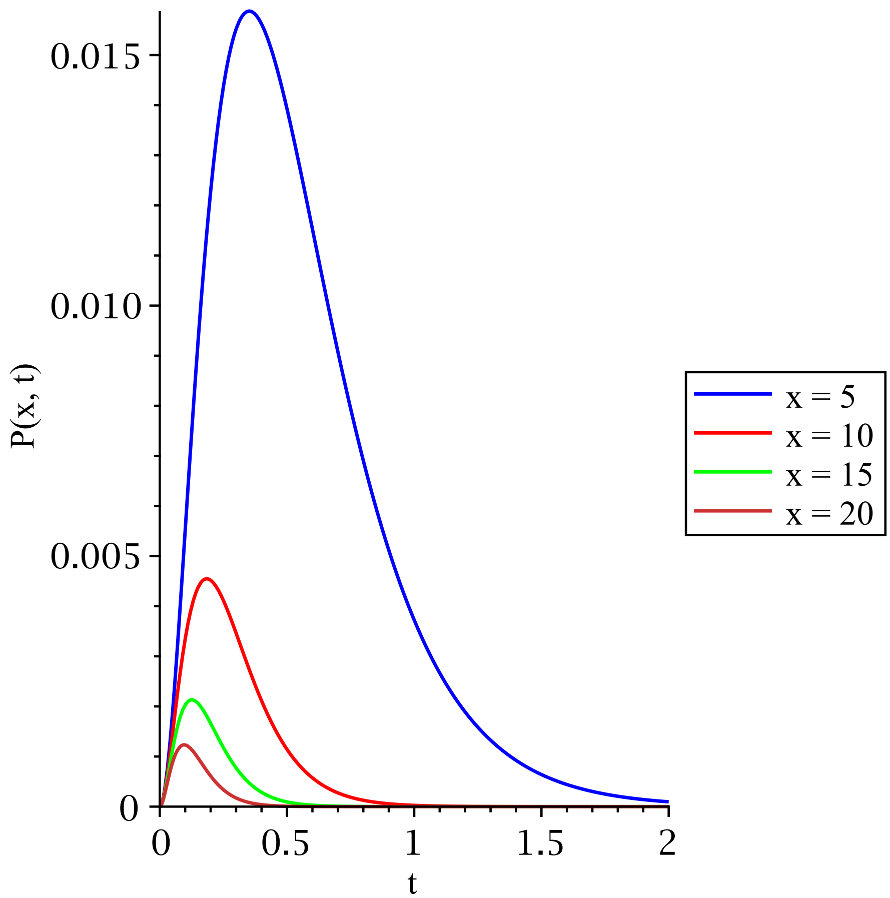

Figure 4.5: The gap density function

In figure 4.5 we see the behaviour of the gap density function for a range of values of $x \geq 1$. As can be seen, as t increases, the density of these larger gaps increases sharply to a maximum, before decreasing, and approaching 0. This is to be expected due to the fact that, as unit cars are initially parked they leave large gaps which are eventually destroyed in the parking process.

4.3 Remarks

With the kinetic approach we have successfully modelled the process with respect to it's time evolution, rather than merely providing a means of evaluating C_R. This is a big step forward from the previous approaches.

It should be noted that the approach of reducing the rate equation to an ODE breaks down for finite L:

$$\begin{aligned}
\frac{\partial}{\partial t}(A(t)e^{-(x-1)t}) &= -(x-1)A(t)e^{-(x-1)t} + 2\int_{x+1}^{L} A(t)e^{-(y-1)t}dy \\
A'(t)e^{-(x-1)t} - (x-1)A(t)e^{-(x-1)t} &= -(x-1)A(t)e^{-(x-1)t} + 2A(t)\left.\frac{e^{-(y-1)t}}{-t}\right|_{y=x+1}^{L} \\
A'(t)e^{-(x-1)t} &= 2A(t)\left[-\frac{e^{-t}}{t}e^{-(L-2)t} + \frac{e^{-t}}{t}e^{-(x-1)t}\right] \\
&= 2A(t)\frac{e^{-t}}{t}\left[e^{-(x-1)t} - e^{-(L-2)t}\right]
\end{aligned}$$

Hence finite L prevents us from reducing the equation to a more manageable ODE, and it is clear that the imposition of $L \to \infty$ is made for mathematical reasons.

Chapter 5

Generalizations

5.1 Parking with overlap

We now look at the problem of parking with overlap, which is of practical interest in the context of particle adsorption (see [4]). Let ϕ denote the region of overlap, with $0 \leq \phi \leq 1$. We therefore have an exclusion zone around the centre of a car of length $1-\phi$. Our problem is now the process of finding parking spots where exclusion zones do not overlap. This is equivalent to the parking problem but instead of dealing with cars of unit length we park the exclusion zones of length $1-\phi$. We introduce a new gap density function $P_\phi(x,t)$ taking this into account, with allowed overlap ϕ.

Let $\bar{x}$ denote the distance between cars, and x denote the distance between exclusion zones. Clearly $\bar{x} = x - \phi$. From this we can see that the distance between exclusion zones is always positive, but the distance between cars can be negative, in the case of overlap.

Similar to the Kinetic approach, we define a rate equation for the creation and destruction of gaps for the overlap case:

$$\frac{\partial P_\phi(x,t)}{\partial t} = \begin{cases} 2\int_{x+1-\phi}^{\infty} P_\phi(y,t)dy & \text{for } x < 1-\phi \\ \\ -(x-1+\phi)P_\phi(x,t) + 2\int_{x+1-\phi}^{\infty} P_\phi(y,t)dy & \text{for } x \geq 1-\phi \end{cases}$$

We take a similar approach to solving the above as was taken in the Kinetic approach. i.e. if we let:

$$P_\phi(x,t) = A(t)e^{-(x-1+\phi)t}$$

and make the further substitution $A(t) = t^2 F_\phi(t)$, we find that:

$$F_\phi(t) = \exp\left(-2\int_0^t \frac{(1-e^{-(1-\phi)\tau})}{\tau} d\tau\right)$$

The gap density function for gaps between exclusion zones becomes:

$$P_\phi(x,t) = \begin{cases} 2\int_0^t \tau F_\phi(\tau)e^{-x\tau}d\tau & \text{for } x < 1-\phi \\ \\ t^2 F_\phi(t)e^{-(x-1+\phi)t} & \text{for } x \geq 1-\phi \end{cases}$$

and coverage by exclusion zone becomes:

$$\frac{d\theta_\phi}{dt} = (1-\phi)F_\phi(t)$$

$$\therefore \quad \theta_\phi(t) \quad = \quad (1-\phi)\int_0^t F_\phi(\tau)d\tau$$

In figure 5.1 we see the behaviour of the coverage function for different values of ϕ. As is to be expected, the coverage tends towards C_R as t increases.

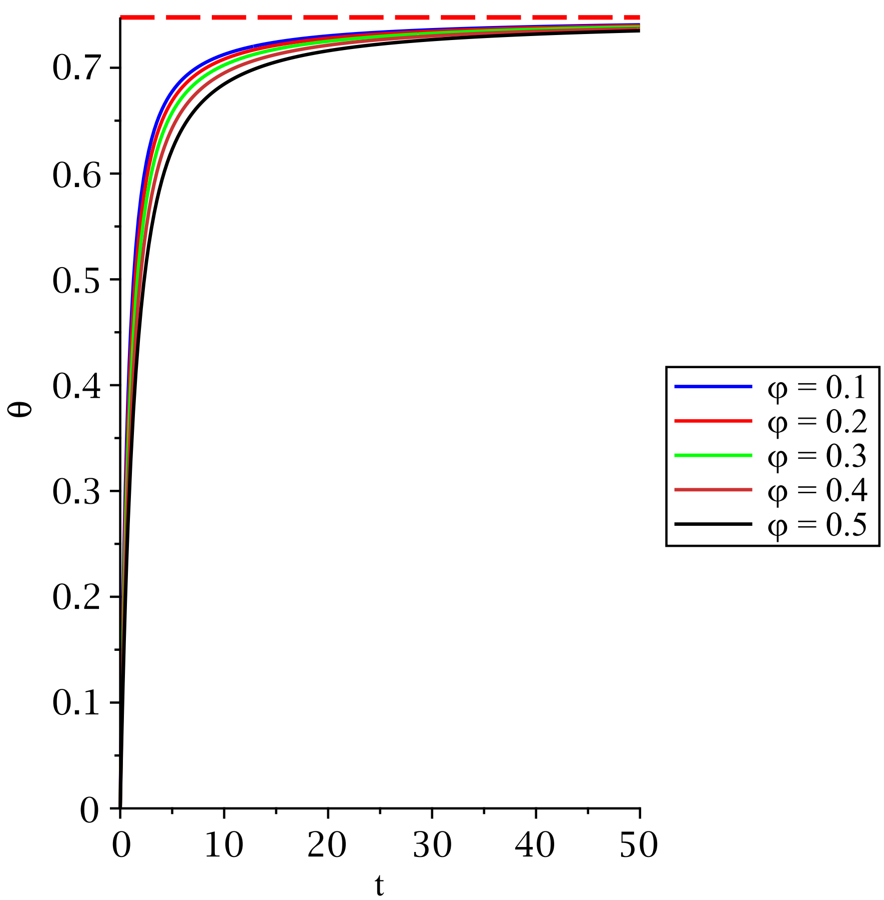

Figure 5.1: The coverage function - overlap case

The gap density function for gaps between cars, $\bar{x} = x - \phi$, becomes:

$$Q_\phi(\bar{x},t) = \begin{cases} 2\int_0^t \tau F_\phi(\tau) e^{-(\bar{x}+\phi)\tau} d\tau & \text{for } \bar{x} < 1-2\phi \\ \\ t^2 F_\phi(t) e^{-(\bar{x}-1+2\phi)t} & \text{for } \bar{x} \geq 1-2\phi \end{cases}$$

and coverage by cars becomes:

$$\Theta_\phi(t) = 1 - \int_0^\infty \bar{x} Q_\phi(\bar{x},t) d\bar{x}$$

Differentiating with respect to t we get:

$$\begin{aligned}
\frac{d\Theta_\phi}{dt} &= -\frac{\partial}{\partial t}\int_0^\infty \bar{x} Q_\phi(\bar{x},t) d\bar{x} \\
&= -\frac{\partial}{\partial t}\left(\int_0^{1-2\phi} \bar{x} Q_\phi(\bar{x},t) d\bar{x} - \int_{1-2\phi}^\infty \bar{x} Q_\phi(\bar{x},t) d\bar{x}\right) \\
&= -\frac{\partial}{\partial t}\left(\int_0^{1-2\phi} \bar{x}\left(2\int_0^t \tau F_\phi(\tau) e^{-(\bar{x}+\phi)\tau} d\tau\right) d\bar{x} - \int_{1-2\phi}^\infty \bar{x} t^2 F_\phi(t) e^{-(\bar{x}-1+2\phi)t} d\bar{x}\right) \\
&= -\frac{\partial}{\partial t}\int_0^{1-2\phi} \bar{x}\left(2\int_0^t \tau F_\phi(\tau) e^{-(\bar{x}+\phi)\tau} d\tau\right) d\bar{x} - \frac{\partial}{\partial t}\int_{1-2\phi}^\infty \bar{x} t^2 F_\phi(t) e^{-(\bar{x}-1+2\phi)t} d\bar{x}
\end{aligned}$$

for $\phi < \frac{1}{2}$. We will deal with each part of the right hand side separately. Starting with the first part:

$$-\frac{\partial}{\partial t}\int_0^{1-2\phi} \bar{x}\left(2\int_0^t \tau F_\phi(\tau) e^{-(\bar{x}+\phi)\tau} d\tau\right) d\bar{x} \quad = \quad -\int_0^{1-2\phi} \bar{x}\left(2\frac{\partial}{\partial t}\int_0^t \tau F_\phi(\tau) e^{-(\bar{x}+\phi)\tau} d\tau\right) d\bar{x}$$

$$
\begin{aligned}
&= -\int_0^{1-2\phi} \bar{x}2tF_\phi(t)e^{-(\bar{x}+\phi)t}d\bar{x} \\
&= -2tF_\phi(t)e^{-\phi t}\int_0^{1-2\phi} \bar{x}e^{-\bar{x}t}d\bar{x} \\
&= -2tF_\phi(t)e^{-\phi t}\left[-\frac{(1-2\phi)e^{-(1-2\phi)t}}{t} - \frac{e^{-(1-2\phi)t}}{t^2} + \frac{1}{t^2}\right] \\
&= 2F_\phi(t)(1-2\phi)e^{-(1-\phi)t} + \frac{2F_\phi(t)e^{-(1-\phi)t}}{t} - \frac{2F_\phi(t)e^{-\phi t}}{t}
\end{aligned}
$$

moving on to the second part:

$$
\begin{aligned}
-\frac{\partial}{\partial t}\int_{1-2\phi}^{\infty} \bar{x}t^2F_\phi(t)e^{-(\bar{x}-1+2\phi)t}d\bar{x} &= -\int_{1-2\phi}^{\infty} \bar{x}\frac{\partial}{\partial t}\left(t^2F_\phi(t)e^{-(\bar{x}-1+2\phi)t}\right)d\bar{x} \\
&= -\int_{1-2\phi}^{\infty} \bar{x}\frac{\partial}{\partial t}\left(t^2F_\phi(t)e^{(1-2\phi)t}e^{-\bar{x}t}\right)d\bar{x}
\end{aligned}
$$

The differential within the integral is:

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(t^2F_\phi(t)e^{(1-2\phi)t}e^{-\bar{x}t}\right) &= 2tF_\phi(t)e^{(1-2\phi)t}e^{-\bar{x}t} \\
&\quad +t^2F'_\phi(t)e^{(1-2\phi)t}e^{-\bar{x}t} \\
&\quad +t^2F_\phi(t)(1-2\phi)e^{(1-2\phi)t}e^{-\bar{x}t} \\
&\quad -t^2F_\phi(t)e^{(1-2\phi)t}\bar{x}e^{-\bar{x}t}
\end{aligned}
$$

so our integral becomes:

$$-\int_{1-2\phi}^{\infty} \bar{x}\frac{\partial}{\partial t}\left(t^2 F_\phi(t) e^{(1-2\phi)t} e^{-\bar{x}t}\right) d\bar{x} = -\int_{1-2\phi}^{\infty} \bar{x} 2t F_\phi(t) e^{(1-2\phi)t} e^{-\bar{x}t} d\bar{x}$$

$$-\int_{1-2\phi}^{\infty} \bar{x} t^2 F'_\phi(t) e^{(1-2\phi)t} e^{-\bar{x}t} d\bar{x}$$

$$-\int_{1-2\phi}^{\infty} \bar{x} t^2 F_\phi(t)(1-2\phi) e^{(1-2\phi)t} e^{-\bar{x}t} d\bar{x}$$

$$+\int_{1-2\phi}^{\infty} \bar{x} t^2 F_\phi(t) e^{(1-2\phi)t} \bar{x} e^{-\bar{x}t} d\bar{x}$$

taking each integral separately:

$$\begin{aligned}
-\int_{1-2\phi}^{\infty} \bar{x} 2t F_\phi(t) e^{(1-2\phi)t} e^{-\bar{x}t} d\bar{x} &= -\int_{1-2\phi}^{\infty} 2t F_\phi(t) e^{(1-2\phi)t} \bar{x} e^{-\bar{x}t} d\bar{x} \\
&= -2t F_\phi(t) e^{(1-2\phi)t} \int_{1-2\phi}^{\infty} \bar{x} e^{-\bar{x}t} d\bar{x} \\
&= -2t F_\phi(t) e^{(1-2\phi)t} \left[\frac{(1-2\phi)e^{-(1-2\phi)t}}{t} + \frac{e^{-(1-2\phi)t}}{t^2}\right] \\
&= -2t F_\phi(t)(1-2\phi) - \frac{2F_\phi(t)}{t}
\end{aligned}$$

followed by:

$$-\int_{1-2\phi}^{\infty} \bar{x} t^2 F'_\phi(t) e^{(1-2\phi)t} e^{-\bar{x}t} d\bar{x} = -\int_{1-2\phi}^{\infty} t^2 F'_\phi(t) e^{(1-2\phi)t} \bar{x} e^{-\bar{x}t} d\bar{x}$$

$$
\begin{aligned}
&= -t^2 F'_\phi(t) e^{(1-2\phi)t} \int_{1-2\phi}^{\infty} \bar{x} e^{-\bar{x}t} d\bar{x} \\
&= -t^2 F'_\phi(t) e^{(1-2\phi)t} \left[\frac{(1-2\phi)e^{-(1-2\phi)t}}{t} + \frac{e^{-(1-2\phi)t}}{t^2} \right] \\
&= -tF'_\phi(t)(1-2\phi) - F'_\phi \\
&= -tF_\phi(t) \left(-2\frac{(1-e^{-(1-\phi)t})}{t} \right)(1-2\phi) - F_\phi(t)\left(-2\frac{(1-e^{-(1-\phi)t})}{t} \right) \\
&= 2F_\phi(t)(1-2\phi) - 2F_\phi(t)(1-2\phi)e^{-(1-\phi)t} + \frac{2F_\phi(t)}{t} - \frac{2F_\phi(t)e^{-(1-\phi)t}}{t}
\end{aligned}
$$

here we make use of the fact that:

$$
\begin{aligned}
F_\phi(t) &= \exp\left(-2\int_0^t \frac{(1-e^{-(1-\phi)\tau})}{\tau} d\tau \right) \\
\therefore \quad F'_\phi(t) &= F_\phi(t)\left(-2\frac{(1-e^{-(1-\phi)t})}{t} \right)
\end{aligned}
$$

followed by:

$$
\begin{aligned}
-\int_{1-2\phi}^{\infty} \bar{x} t^2 F_\phi(t)(1-2\phi)e^{(1-2\phi)t} e^{-\bar{x}t} d\bar{x} &= -\int_{1-2\phi}^{\infty} t^2 F_\phi(t)(1-2\phi)e^{(1-2\phi)t} \bar{x} e^{-\bar{x}t} d\bar{x} \\
&= -t^2 F_\phi(t)(1-2\phi)e^{(1-2\phi)t} \int_{1-2\phi}^{\infty} \bar{x} e^{-\bar{x}t} d\bar{x} \\
&= -t^2 F_\phi(t)(1-2\phi)e^{(1-2\phi)t} \left[\frac{(1-2\phi)e^{-(1-2\phi)t}}{t} + \frac{e^{-(1-2\phi)t}}{t^2} \right]
\end{aligned}
$$

$$= \quad -tF_{\phi}(t)(1-2\phi)^2 - F_{\phi}(t)(1-2\phi)$$

and finally:

$$\begin{aligned}
\int_{1-2\phi}^{\infty} \bar{x} t^2 F_{\phi}(t) e^{(1-2\phi)t} \bar{x} e^{-\bar{x}t} d\bar{x} &= \int_{1-2\phi}^{\infty} t^2 F_{\phi}(t) e^{(1-2\phi)t} \bar{x}^2 e^{-\bar{x}t} d\bar{x} \\
&= t^2 F_{\phi}(t) e^{(1-2\phi)t} \int_{1-2\phi}^{\infty} \bar{x}^2 e^{-\bar{x}t} d\bar{x} \\
&= t^2 F_{\phi}(t) e^{(1-2\phi)t} \left[\frac{(1-2\phi)^2 e^{-(1-2\phi)t}}{t} + \frac{2(1-2\phi) e^{-(1-2\phi)t}}{t^2} + \frac{2e^{-(1-2\phi)t}}{t^3} \right] \\
&= tF_{\phi}(t)(1-2\phi)^2 + 2F_{\phi}(t)(1-2\phi) + \frac{2F_{\phi}(t)}{t}
\end{aligned}$$

combining all of the above we get:

$$\begin{aligned}
\frac{d\Theta_{\phi}}{dt} &= 2F_{\phi}(t)(1-2\phi)e^{-(1-\phi)t} + \frac{2F_{\phi}(t)e^{-(1-\phi)t}}{t} - \frac{2F_{\phi}(t)e^{-\phi t}}{t} \\
& -2tF_{\phi}(t)(1-2\phi) - \frac{2F_{\phi}(t)}{t} \\
& +2F_{\phi}(t)(1-2\phi) - 2F_{\phi}(t)(1-2\phi)e^{-(1-\phi)t} + \frac{2F_{\phi}(t)}{t} - \frac{2F_{\phi}(t)e^{-(1-\phi)t}}{t} \\
& -tF_{\phi}(t)(1-2\phi)^2 - F_{\phi}(t)(1-2\phi) \\
& +tF_{\phi}(t)(1-2\phi)^2 + 2F_{\phi}(t)(1-2\phi) + \frac{2F_{\phi}(t)}{t}
\end{aligned}$$

after much cancellation, we are left with:

$$\begin{aligned}\frac{d\Theta_\phi}{dt} &= \frac{2F_\phi(t)}{t} - \frac{2F_\phi(t)e^{-\phi t}}{t} + F_\phi(t)(1-2\phi) \\ &= F_\phi(t)\left[\frac{2}{t}(1-e^{-\phi t}) + 1 - 2\phi\right]\end{aligned}$$

and hence:

$$\Theta_\phi(t) = (1-2\phi)\int_0^t F_\phi(\tau)d\tau + \int_0^t F_\phi(\tau)\frac{2}{\tau}(1-e^{-\phi\tau})d\tau$$

for $\phi < \frac{1}{2}$. Now similarly for the case when $\phi \geq \frac{1}{2}$:

$$\begin{aligned}\frac{d\Theta_\phi}{dt} &= -\frac{\partial}{\partial t}\int_0^\infty \bar{x}Q_\phi(\bar{x},t)d\bar{x} \\ &= -\int_0^\infty \bar{x}\frac{\partial}{\partial t}\left(t^2F_\phi(t)e^{(1-2\phi)t}e^{-\bar{x}t}\right)d\bar{x} \\ &= -2tF_\phi(t)e^{(1-2\phi)t}\int_0^\infty \bar{x}e^{-\bar{x}t}d\bar{x} \\ &\quad -t^2F'_\phi(t)e^{(1-2\phi)t}\int_0^\infty \bar{x}e^{-\bar{x}t}d\bar{x} \\ &\quad -t^2F_\phi(t)(1-2\phi)e^{(1-2\phi)t}\int_0^\infty \bar{x}e^{-\bar{x}t}d\bar{x} \\ &\quad +t^2F_\phi(t)e^{(1-2\phi)t}\int_0^\infty \bar{x}^2e^{-\bar{x}t}d\bar{x}\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{t}F_\phi(t)e^{(1-2\phi)t} - F'_\phi(t)e^{(1-2\phi)t} - F_\phi(t)(1-2\phi)e^{(1-2\phi)t} + \frac{2}{t}F_\phi(t)e^{(1-2\phi)t} \\
&= -F'_\phi(t)e^{(1-2\phi)t} - F_\phi(t)(1-2\phi)e^{(1-2\phi)t} \\
&= -\frac{d}{dt}\left(F_\phi(t)e^{(1-2\phi)t}\right) \\
\therefore \quad \Theta_\phi(t) &= -F_\phi(t)e^{(1-2\phi)t} + C
\end{aligned}
$$

at $t = 0$ the coverage is 0, and hence $C = 1$, giving us:

$$
\Theta_\phi(t) = 1 - F_\phi(t)e^{(1-2\phi)t}
$$

so our coverage for cars is:

$$
\Theta_\phi(t) = \begin{cases} (1-2\phi)\int_0^t F_\phi(\tau)d\tau + \int_0^t F_\phi(\tau)\frac{2}{\tau}(1-e^{-\phi\tau})d\tau & \text{for } \phi < \frac{1}{2} \\ \\ 1 - F_\phi(t)e^{(1-2\phi)t} & \text{for } \phi \geq \frac{1}{2} \end{cases}
$$

In figure 5.2 we see the behaviour of the coverage for cars for values of ϕ between 0 and 0.5. As is to be expected, the limit for each value of ϕ increases as t increases.

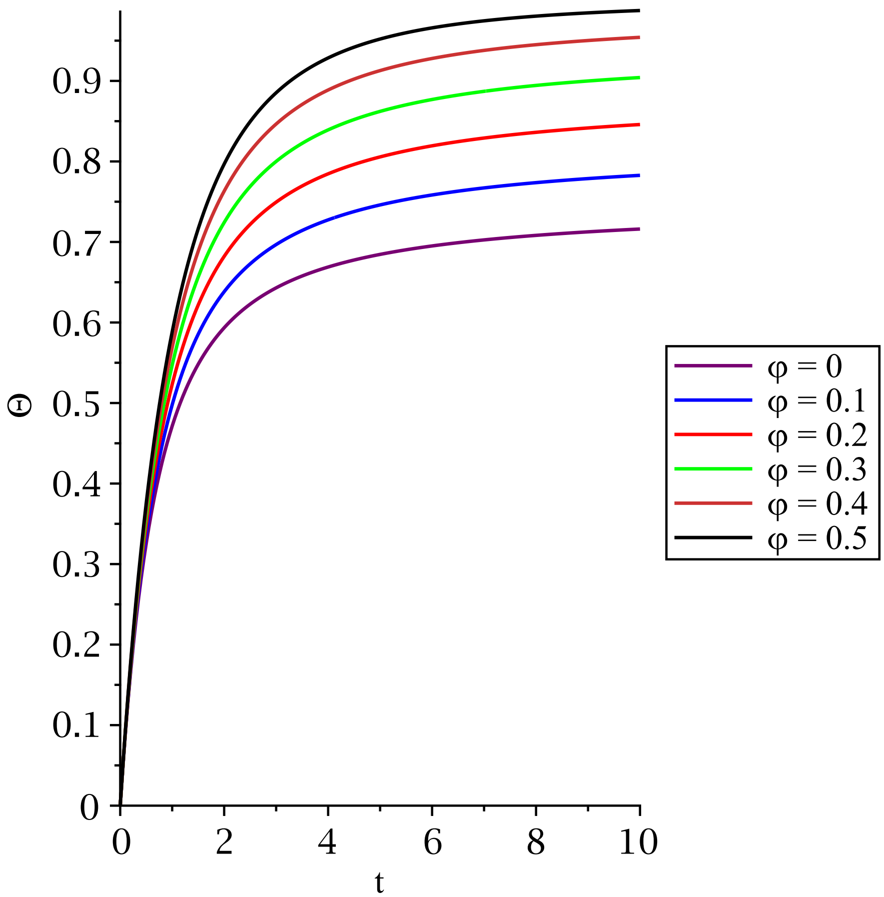

Figure 5.2: The coverage by cars

In figure 5.3 we see the behaviour of the coverage for cars for values of ϕ between 0.5 and 1. As is to be expected, the coverage tends towards 1 as t increases.

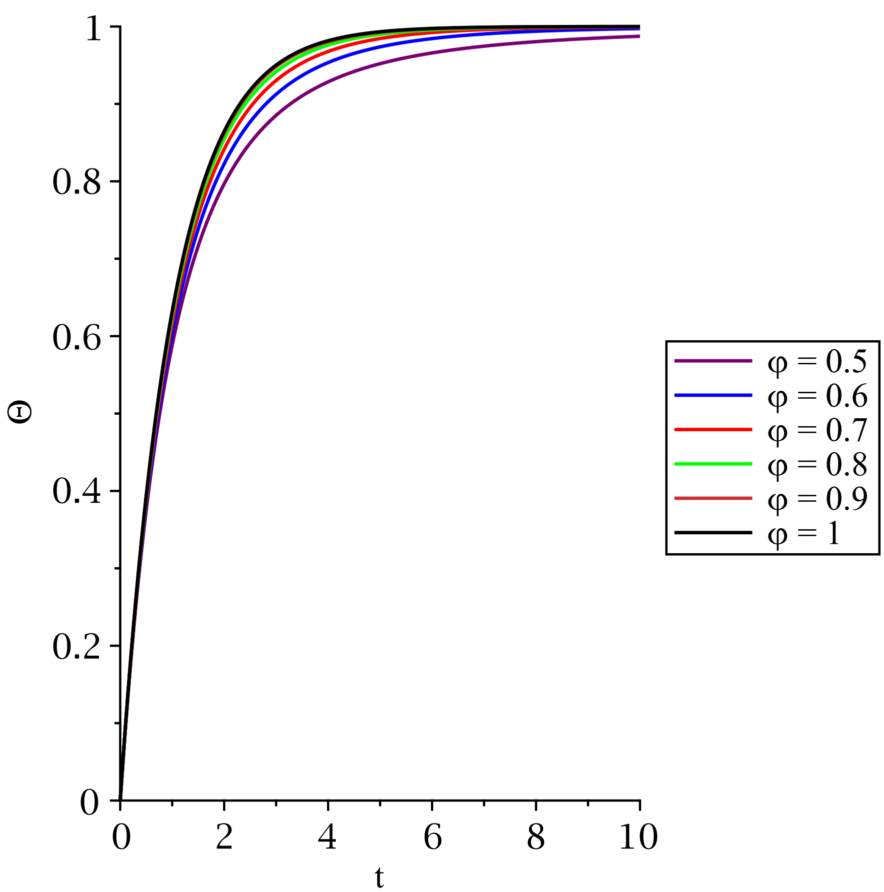

Figure 5.3: The coverage by cars

We now consider the car coverage for $\phi = 1$. We first consider $F_\phi(t)$ when $\phi = 1$:

$$\begin{aligned}
F_{\phi=1}(t) &= \exp\left(-2\int_0^t \frac{(1-e^0)}{\tau}d\tau\right) \\
&= \exp\left(-2\int_0^t \frac{(1-1)}{\tau}d\tau\right) \\
&= \exp\left(-2\int_0^t \frac{0}{\tau}d\tau\right) \\
&= \exp(0) \\
&= 1
\end{aligned}$$

making use of this we find:

$$\begin{aligned}
\Theta_{\phi=1}(t) &= 1 - F_{\phi=1}(t)e^{(1-2)t} \\
&= 1 - e^{-t} \\
\lim_{t\to\infty}\Theta_{\phi=1}(t) &= 1 - \lim_{t\to\infty} e^{-t} \\
&= 1
\end{aligned}$$

We now consider the car coverage for $\frac{1}{2} < \phi < 1$:

$$\begin{aligned}
\lim_{t\to\infty}\Theta_\phi(t) &= 1-\lim_{t\to\infty}F_\phi(t)e^{(1-2\phi)t} \\
&= 1-\lim_{t\to\infty}F_\phi(t)\cdot\lim_{t\to\infty}e^{-(2\phi-1)t} \\
&= 1-\lim_{t\to\infty}F_\phi(t)\cdot 0 \\
&= 1
\end{aligned}$$

We now consider the car coverage for $\phi = \frac{1}{2}$. We first consider $F_\phi(t)$ when $\phi = \frac{1}{2}$:

$$\begin{aligned}
F_{\phi=1/2}(t) &= \exp\left(-2\int_0^t \frac{(1-e^{-\tau/2})}{\tau}d\tau\right) \\
&= \exp\left(-2\int_0^t \frac{1}{\tau}d\tau + 2\int_0^t \frac{e^{-\tau/2}}{\tau}d\tau\right) \\
&= \exp\left(-2\int_0^t \frac{1}{\tau}d\tau\right)\cdot\exp\left(2\int_0^t \frac{e^{-\tau/2}}{\tau}d\tau\right) \\
&= \exp\left(-2\ \ln(\tau)\big|_{\tau=0}^{t}\right)\cdot\exp\left(2\int_0^t \frac{e^{-\tau/2}}{\tau}d\tau\right) \\
&= \exp\left(-\ \ln(\tau^2)\big|_{\tau=0}^{t}\right)\cdot\exp\left(2\int_0^t \frac{e^{-\tau/2}}{\tau}d\tau\right) \\
&= \exp\left(\ln\left(\frac{0}{t^2}\right)\right)\cdot\exp\left(2\int_0^t \frac{e^{-\tau/2}}{\tau}d\tau\right) \\
&= \frac{0}{t^2}\cdot\exp\left(2\int_0^t \frac{e^{-\tau/2}}{\tau}d\tau\right)
\end{aligned}$$

$$\therefore \quad F_{\phi=1/2}(t) = 0$$

and so we have:

$$\begin{aligned} \Theta_{\phi=1/2}(t) &= 1 - F_{\phi=1/2}(t) \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

hence if ϕ is at least $\frac{1}{2}$, then full coverage is achieved. In figure 5.4 we see the behaviour of the coverage for cars for values of ϕ between 0 and 1.

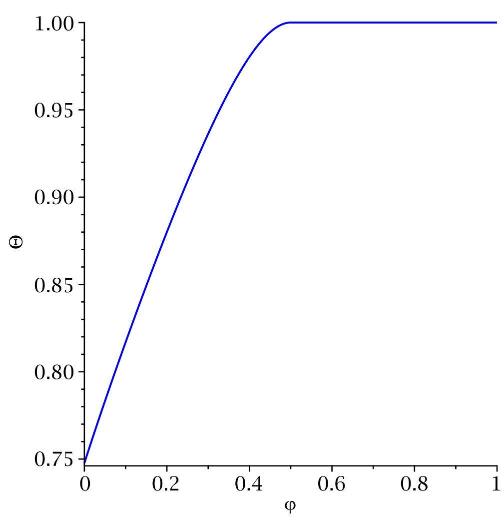

Figure 5.4: The coverage function as $t \to \infty$

5.2 The reversible parking problem

We now look at the reversible parking problem, where cars can both park and leave (see [2]). In this version of the problem we assume that cars park at a rate of k_+ and leave at a rate of k_-. Thus far we have only considered the irreversible problem, which stops once the jamming limit has been reached. With the reversible problem we reach an equilibrium state when a balance is reached between the arrival rate k_+ and the leaving rate k_-. We define our rate equations for this case as follows, beginning with the $x < 1$ case:

$$\frac{\partial P(x,t)}{\partial t} = -2k_- P(x,t) + 2k_+ \int_{x+1}^{\infty} P(y,t)dy$$

in which we have an adsorption (+) and a desorption (−) term. The adsorption term describes when a car parks within an interval of size $y > x + 1$, leaving a gap of length x, and this it can do in two different ways. The desorption term describes how a gap can be destroyed when any one of it's two neighbouring cars leaves.

For the $x \geq 1$ case:

$$\begin{aligned}\frac{\partial P(x,t)}{\partial t} &= -2k_- P(x,t) + 2k_+ \int_{x+1}^{\infty} P(y,t)dy \\ &\quad -k_+(x-1)P(x,t) + \frac{k_-}{\theta(t)} \int_{0}^{x-1} P(y,t)P(x-1-y,t)dy\end{aligned}$$

we have an additional two terms. The first describes how gaps of length greater than a car size are destroyed by arriving cars, with a rate $k_+(x-1)$. The final term describes how two gaps separated by a car can form a larger gap when that car leaves. This term can be defined in terms of a gap-gap density function $P(y,z,t)$ which defines the density of neighbouring gaps of size y and z separated by a car. It can be defined as follows:

$$P(y,z,t) = \frac{P(y,t)P(z,t)}{\theta(t)}$$

In defining the gap-gap density function, we assume independence of the gap density functions, so that the distribution of two gaps separated by a single car is proportional to the product of the individual gap distributions. The coverage term in the denominator provides normalization. So the gap-gap density term above deals with the case where two neighbouring gaps, of size y and $x-1-y$, separated by a single car, form a gap of size x if the car separating the gaps leaves.

From the above, it is clear that our condition for equilibrium is:

$$\frac{\partial P(x,t)}{\partial t} = 0$$

using the above rate equation we get:

$$\begin{aligned} -2k_{-}P(x) + 2k_{+}\int_{x+1}^{\infty} P(y)dy &= 0 \\ 2k_{+}\int_{x+1}^{\infty} P(y)dy &= 2k_{-}P(x) \\ \int_{x+1}^{\infty} P(y)dy &= \frac{k_{-}}{k_{+}}P(x) \end{aligned}$$

using the substitution $P(x) = \beta e^{-\alpha x}$ we get:

$$\int_{x+1}^{\infty} \beta e^{-\alpha y} dy = \frac{k_-}{k_+} \beta e^{-\alpha x}$$

$$\beta \int_{x+1}^{\infty} e^{-\alpha y} dy = \frac{k_-}{k_+} \beta e^{-\alpha x}$$

$$\beta \left. \frac{e^{-\alpha y}}{-\alpha} \right|_{y=x+1}^{\infty} = \frac{k_-}{k_+} \beta e^{-\alpha x}$$

$$\frac{e^{-\alpha}}{\alpha} \beta e^{-\alpha x} = \frac{k_-}{k_+} \beta e^{-\alpha x}$$

$$\frac{e^{-\alpha}}{\alpha} = \frac{k_-}{k_+}$$

$$\alpha e^{\alpha} = \frac{k_+}{k_-}$$

returning to our total coverage equation, and making the same substitution:

$$\int_0^{\infty} (x+1) P(x) dx = 1$$

$$\int_0^{\infty} x P(x) dx + \int_0^{\infty} P(x) dx = 1$$

$$\int_0^{\infty} x \beta e^{-\alpha x} dx + \int_0^{\infty} \beta e^{-\alpha x} dx = 1$$

$$\beta \int_0^{\infty} x e^{-\alpha x} dx + \beta \int_0^{\infty} e^{-\alpha x} dx = 1$$

$$\begin{aligned} \beta\left(\frac{1}{\alpha^2}+\frac{1}{\alpha}\right) &= 1 \\ \beta(1+\alpha) &= \alpha^2 \\ \beta &= \frac{\alpha^2}{1+\alpha} \end{aligned}$$

which gives us for our equilibrium gap density function:

$$P_{eq}(x) = \frac{\alpha^2}{1+\alpha}e^{-\alpha x}$$

We look next at the equilibrium coverage θ_{eq}:

$$\begin{aligned} \theta_{eq} &= \int_0^\infty P_{eq}(x)dx \\ &= \int_0^\infty \frac{\alpha^2}{1+\alpha}e^{-\alpha x}dx \\ &= \frac{\alpha^2}{1+\alpha}\int_0^\infty e^{-\alpha x}dx \\ &= \frac{\alpha^2}{1+\alpha}\cdot\frac{1}{\alpha} \\ &= \frac{\alpha}{1+\alpha} \end{aligned}$$

And next we look at the function $\alpha \to \alpha e^\alpha$ on $(0,\infty)$:

$$f(\alpha) = \alpha e^\alpha$$

$$
\begin{aligned}
f'(\alpha) &= \alpha e^{\alpha} + e^{\alpha} \\
&= (\alpha + 1)e^{\alpha} \\
&> 0
\end{aligned}
$$

hence $f(\alpha)$ is strictly increasing on $(0, \infty)$. Therefore, if we were to graph the functions $y = f(\alpha)$ and $y = \frac{k_+}{k_-}$, they would meet in one and only one place, and hence there is a unique solution for $\alpha e^{\alpha} = \frac{k_+}{k_-}$. And also, as $\alpha e^{\alpha} = \frac{k_+}{k_-}$, if $\frac{k_+}{k_-} \to \infty$ then $\alpha e^{\alpha} \to \infty$ which implies $\alpha \to \infty$. In order to plot the equilibrium coverage as a function of $\frac{k_+}{k_-}$, we must first perform some manipulation:

$$
\begin{aligned}
\alpha e^{\alpha} &= \frac{k_+}{k_-} \\
\ln(\alpha e^{\alpha}) &= \ln\left(\frac{k_+}{k_-}\right) \\
\ln(\alpha) + \alpha &= \ln\left(\frac{k_+}{k_-}\right)
\end{aligned}
$$

Clearly, α dominates the left hand side as $\alpha \to \infty$, and hence:

$$
\alpha \approx \ln\left(\frac{k_+}{k_-}\right)
$$

putting this approximation for α into our expression for θ_{eq}:

$$\begin{aligned}\theta_{eq} &= \frac{\alpha}{1+\alpha}\\ &\approx \frac{\alpha-1}{\alpha} \quad \text{as } \alpha \to \infty\\ &\approx \frac{\ln(k_+/k_-)-1}{\ln(k_+/k_-)}\\ &\approx 1-\frac{1}{\ln(k_+/k_-)}\end{aligned}$$

So we now have an approximation for the equilibrium coverage function as a function of $\frac{k_+}{k_-}$. In figure 5.5 we see the behaviour of the equilibrium coverage function. We can see that θ_{eq} crosses C_R (the red dashed line) as $\frac{k_+}{k_-}$ reaches approximately 50.

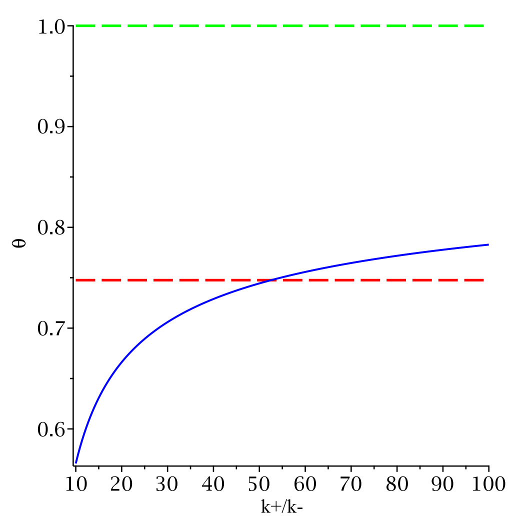

Figure 5.5: The equilibrium coverage function

In figure 5.6 we see the behaviour of the equilibrium coverage function for a greater range of values of $\frac{k_+}{k_-}$. We can see that θ_{eq} continues to increase, albeit very slowly due to the logarithmic growth of the term involving $\frac{k_+}{k_-}$, towards 1 (the green dashed line).

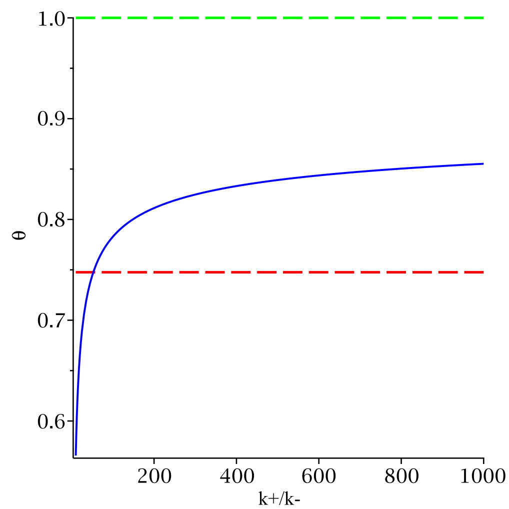

Figure 5.6: The equilibrium coverage function

It is easy to see from our approximation for the equilibrium coverage function that, as $\frac{k_+}{k_-} \to \infty$, $\ln\left(\frac{k_+}{k_-}\right) \to \infty$, and hence:

$$\theta_{eq} \to 1 \quad \text{as } \frac{k_+}{k_-} \to \infty$$

Where $k_- \to 0$, we have $\frac{k_+}{k_-} \to \infty$, and $\theta_{eq} \to 1$. So as k_- gets closer to 0, for example in the case where we can control desorption, the process tends towards full coverage. But in the case where $k_- = 0$, our problem collapses into the standard parking problem, and coverage falls back to C_R.

5.3 Remarks

As with the kinetic approach, we have successfully modelled the process with respect to it's time evolution, and with some generalizations.

These solutions are even more satisfying than the kinetic approach because the assumptions of each extend the assumptions of the kinetic approach:

- **Kinetic Approach:** time evolution of parking cars
- **Overlap Approach:** time evolution of parking exclusion zones
- **Reversible Approach:** time evolution of cars that park and leave at different rates

This can be seen in the construction of each of their rate equations, where the rate equations for each generalization builds on the rate equations of the kinetic approach by adding extra terms reflecting the extra sophistication.

Chapter 6

Simulations

6.1 Introduction

Many of the solutions we have covered in this paper require numerical techniques for evaluation. We present a different approach that serves as both a means of evaluating results, and as a way of confirming theories.

The Monte Carlo approach is a computational technique that is best applied to problems that have a probabilistic element. The technique can be summarized as follows:

- define the range of values to be drawn from
- draw values from this range using a distribution
- perform a computation on the results
- record the results of the computation
- repeat as appropriate

So in the case of the parking problem, the steps would be as follows:

- define the length L on which cars can be parked
- draw values from the interval $(0, L)$ using a distribution to simulate the spots taken by cars parking on the length L
- once no more parking spots remain calculate the coverage, in this case the ratio of cars to the length L
- record the results of this computation
- repeat for as many iterations as required and present statistics of the results of the simulation

The results of the simulation serve as both an estimation of a value, but can also be used to confirm that a theory is valid where experimental verification might be difficult or impractical.

We will initially illustrate the usefulness of this approach by looking at a simple variation on the parking problem, the discrete parking problem (see [3]), where we have an exact result that is easy to evaluate, and demonstrate that the technique produces results that are consistent with the theory.

6.2 The Discrete Parking Problem

In the discrete parking problem we have a length L made up of discrete sites, and we attempt to park cars that occupy two adjacent sites. If a car tries to park at two adjacent empty sites, the process is successful, otherwise unsuccessful. The process continues until there are no more free adjacent empty sites that can accommodate cars.

The coverage for the discrete parking problem is $\theta_d(t)$, and the jamming coverage is:

$$\lim_{t \to \infty} \theta_d(t) = 1 - e^{-2}$$

which is calculated to be 0.864664 to six decimal places. Below we see the output of a simulation for this problem where the number of iterations is set to 10000 and the length L is set to 100000:

```
    Parking Problem - Discrete Version: results

                                    L:   100000
                           iterations:    10000

                         distribution:
                                 mean: 0.864673
                   standard deviation: 0.000848
```

In figure 6.1 we see a plot of the distribution of results from our simulation. It has the familiar bell curve shape which would imply the simulation produces results that are normal, or nearly normal, which in turn tells us, for example, that 95% of the results are within 2 standard deviations of the mean. This in itself should give us confidence in the technique.

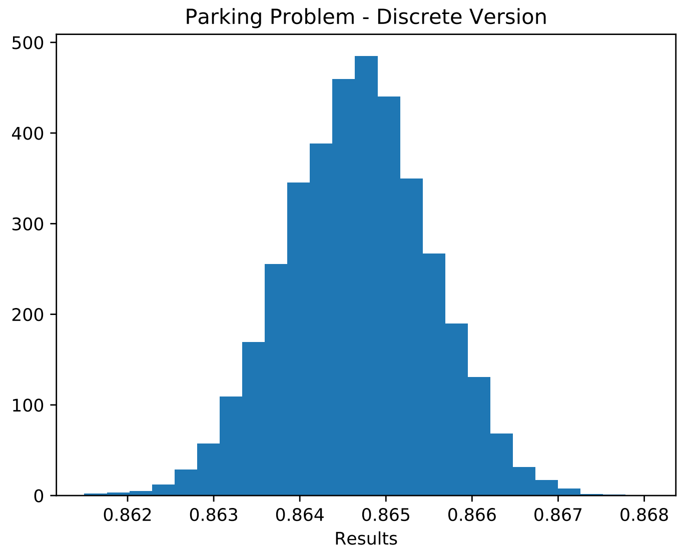

Figure 6.1: Histogram of the parking problem simulation - discrete case

So with not much effort we seem to have arrived at an alternative approach to numerical techniques for evaluation of sometimes complicated results. We see consistency between the result calculated above, and from our simulation, which would seem to validate the mathematics.

6.3 The Parking Problem

Next we will simulate the parking problem. We construct the simulation similarly to the discrete case, except in this case we do not park in discrete adjacent places, so our implementation is actually simpler. The jamming coverage for the parking problem is:

$$C_R = \int_0^\infty \exp\left(-2\int_0^t \frac{1-e^{-u}}{u} du\right) dt$$

which can only be evaluated numerically, and is calculated to be 0.747598 to six decimal places. We see below the output of a simulation for this problem with the number of iterations set to 10000 and L set to 100000:

```
        Parking Problem: results

                     L:    100000
            iterations:     10000

          distribution:
                  mean: 0.747602
    standard deviation: 0.000617
```

In figure 6.2 we see a plot of the distribution of results from our simulation. Once again, we see the familiar bell curve shape which would again imply the simulation produces results that are normal, or nearly normal.

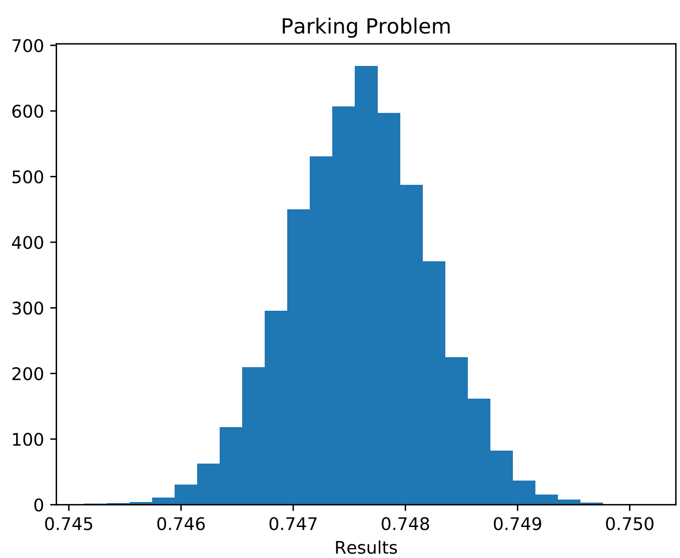

Figure 6.2: Histogram of the parking problem simulation

Once again we see clear consistency between the numerically evaluated result, and the result of the simulation. So again we have provided a pretty successful means of evaluating our constant other than the standard numerical approach.

6.4 Parking with overlap

Next we will simulate parking with overlap. We construct the simulation similarly to the earlier cases, except in this case we park by exclusion zone. The coverage by cars for parking with overlap is:

$$\Theta_\phi(t) = \begin{cases} (1-2\phi)\displaystyle\int_0^t F_\phi(\tau)d\tau + \int_0^t F_\phi(\tau)\frac{2}{\tau}(1-e^{-\phi\tau})d\tau & \text{for } \phi < \dfrac{1}{2} \\ \\ 1 - F_\phi(t)e^{(1-2\phi)t} & \text{for } \phi \geq \dfrac{1}{2} \end{cases}$$

The jamming coverage for a selection of values for ϕ as $t \to \infty$ is shown below in table 6.1:

ϕ	$\lim_{t\to\infty} \Theta_\phi(t)$
0.1	0.816909
0.2	0.880028
0.3	0.936238
0.4	0.980342
0.5	1

Table 6.1: $\lim_{t\to\infty} \Theta_\phi(t)$

As we see, the coverage gets closer to full coverage (i.e. 1) as $\phi \to 0.5$.

We see below the output of a simulation for this problem with the number of iterations set to 10000, L set to 100000, and with ϕ set to 0.1:

```
    Parking Problem - Overlap Version: results

                                   L:   100000
                             overlap:      0.1
                          iterations:    10000

                        distribution:
                                mean: 0.816894
                  standard deviation: 0.000584
```

In figure 6.3 we see a plot of the distribution of results from our simulation for $\phi = 0.1$:

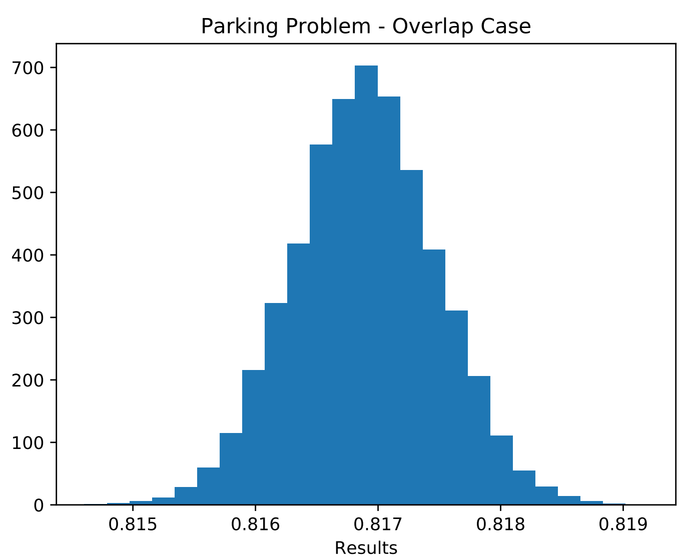

Figure 6.3: Histogram of the parking with overlap simulation - $\phi = 0.1$

We see below the output of a simulation for this problem with the number of iterations set to 10000, L set to 100000, and with ϕ set to 0.2:

```
  Parking Problem - Overlap Version: results

                                 L:   100000
                           overlap:      0.2
                        iterations:    10000

                      distribution:
                              mean: 0.880027
                standard deviation: 0.000477
```

In figure 6.4 we see a plot of the distribution of results from our simulation for $\phi = 0.2$:

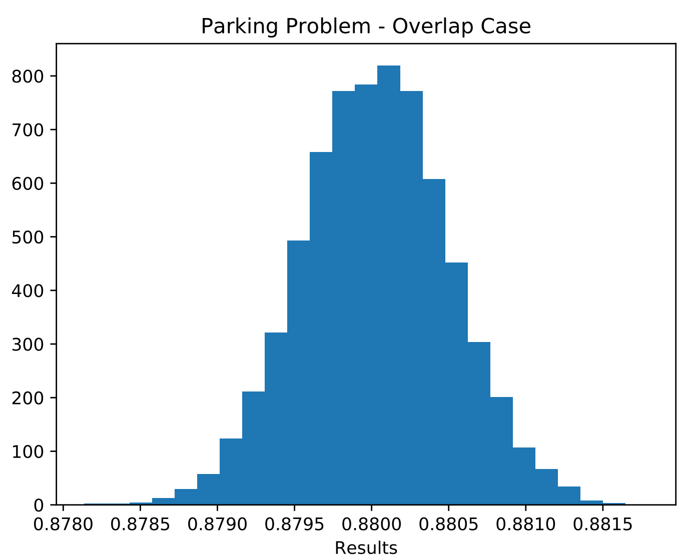

Figure 6.4: Histogram of the parking with overlap simulation - $\phi = 0.2$

We see below the output of a simulation for this problem with the number of iterations set to 10000, L set to 100000, and with ϕ set to 0.3:

```
Parking Problem - Overlap Version: results

                               L:   100000
                         overlap:      0.3
                      iterations:    10000

                    distribution:
                            mean: 0.936235
              standard deviation: 0.000327
```

In figure 6.5 we see a plot of the distribution of results from our simulation for $\phi = 0.3$:

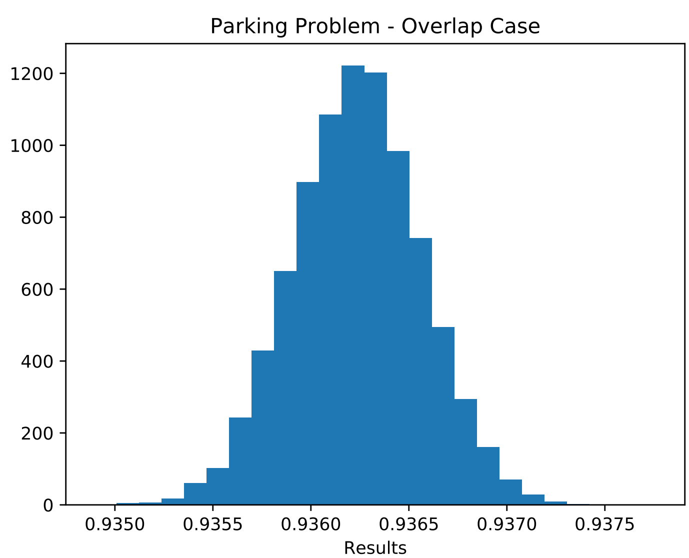

Figure 6.5: Histogram of the parking with overlap simulation - $\phi = 0.3$

We see below the output of a simulation for this problem with the number of iterations set to 10000, L set to 100000, and with ϕ set to 0.4:

```
    Parking Problem - Overlap Version: results

                                     L:   100000
                               overlap:      0.4
                            iterations:    10000

                          distribution:
                                  mean: 0.980344
                    standard deviation: 0.000144
```

In figure 6.6 we see a plot of the distribution of results from our simulation for $\phi = 0.4$:

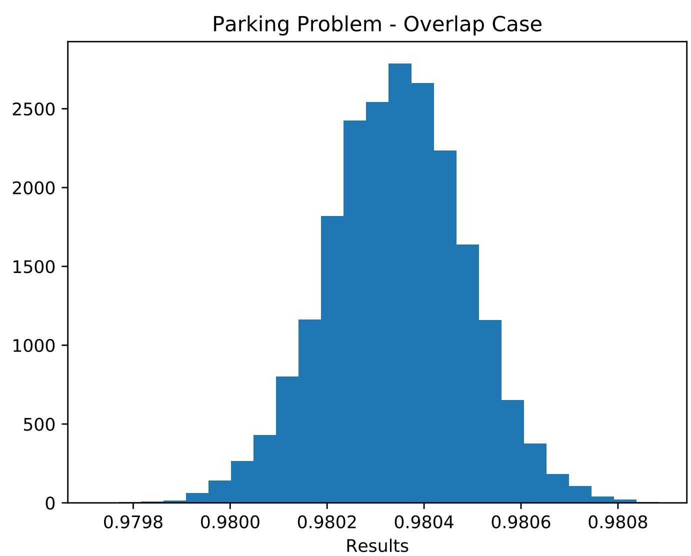

Figure 6.6: Histogram of the parking with overlap simulation - $\phi = 0.4$

We see below the output of a simulation for this problem with the number of iterations set to 10000, L set to 100000, and with ϕ set to 0.5:

```
  Parking Problem - Overlap Version: results

                                 L:   100000
                           overlap:      0.5
                        iterations:    10000

                      distribution:
                              mean: 1.000000
                standard deviation: 0.000000
```

In figure 6.7 we see a plot of the distribution of results from our simulation for $\phi = 0.5$:

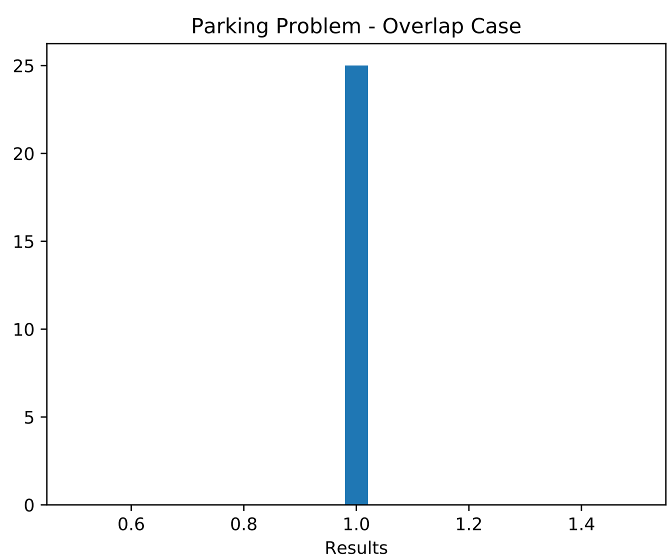

Figure 6.7: Histogram of the parking with overlap simulation - $\phi = 0.5$

So again we see strong consistency between the results of our simulations, and the corresponding calculated results. And the consistent results serve to verify our approach.

6.5 Remarks

There are two approaches to implementing simulations:

- an iterative approach - ordinarily more faithful to the problem, but can take a long time to complete
- a recursive approach - more suited to evaluating results, and completes relatively quickly

In the context of our simulations, if we want to confirm that we have modelled the process correctly, we might choose an iterative implementation, in which case the outcome will be more accurate the more iterations we perform, and for larger values of L, but if the number of iterations is too great, and if L is too large, then the simulation may take a very long time to complete.

So there needs to be a trade-of made:

- if high accuracy of the outcome is required, then you should use lots of iterations, and set L to be as large as possible
- if confirmation of a theory is all that is required, then fewer iterations will be necessary, and a shorter L will suffice
- if the calculation of a value is required, then the recursive approach should be used

Chapter 7

Conclusions

7.1 Remarks

The kinetic approach is wholly satisfying, and leaves the way open to further research. This author was struck by the elegance of the derivations, and spent some time considering whether or not there was some underlying conservation law other than the obvious:

$$\int_0^\infty xP(x,t)dx + \int_0^\infty P(x,t)dx = 1$$

i.e that the normalized sum of the car lengths and gap lengths equals 1. And with respect to the generalizations of the kinetic framing of the problem, once again we see two beautifully elegant solutions. In particular, the reversible problem, and it's condition for equilibrium:

$$\frac{\partial P(x,t)}{\partial t} = 0$$

which would lead one to believe that there are hidden symmetries.

We have demonstrated that the general approach of validating theory through simulations has been useful. When it comes to modelling a problem that does not lend itself easily to empirical verification, such as the parking problem, a simulation can tell you if your theory makes sense very quickly.

Also, we have shown that simulations can help to provide statistical properties relating to the associated constants, which in turn can help establish whether a process is being inhibited by some unknown factor - for example, if an adsorption yield in some industrial process falls outside of 2 standard deviations of the mean yield from a simulation based on the existing conditions, then some investigation might be warranted.

7.2 Future Research

In this author's view there is scope for further research into the derivation of the time-independent limit, in the one-dimensional case, which does not take Rényi's approach. Also, the jamming limit C_R feels like something universal, akin to e or ϕ, or even π. It feels like the surface was only scratched and a much more satisfying number-theoretic approach could be taken. This author would welcome such research.

This author also believes that there may be more interesting results, and possibly hidden symmetries found, from studying this problem in the context of distributions - both in the specific probabilistic, and more general special function, sense.

Higher dimensional problems, in two and three dimensions, would be a logical next research topic - investigating what (if any) relationship exists between two and three dimensional packing problems and C_R.

Sadly, one generalization I did not have time to look at in much depth is *Competitive RSA of a binary mixture*, where cars of two different lengths compete for parking spaces. This is a more complicated problem mathematically, and is actually more difficult to simulate, but might have more general, or practical, applications. This author feels this subject is certainly worthy of more attention.

Appendix A

The Simulation Code

All simulations were implemented in Python. Two implementations were provided for each simulation - a recursive and an iterative implementation. The driver code for each implementation was similar in each case, with the required differences located in one or more extra functions. I will provide listings of the main simulation driver code, and then the different functions that contain the unique implementations for the respective simulations.

In listing A.1 we see the code that runs the simulations and aggregates the results of each simulation.

```
def simulation(iterations, length):
  results = []

  print()
  print(' Parking Problem: running {} simulations'.format(iterations))
  print()

  for i in range(iterations):
    results.append(parking_problem(length))

  print('          L: {:8d}'.format(length))
```

```
  print(' iterations: {:8d}'.format(iterations))
  print()

  return results
```

Listing A.1: Parking Problem - simulation driver

In listing A.2 we see the recursive implementation of the code that parks the cars and calculates the required ratio for the standard case.

```
def parking_problem(length):
  spots = []

  def find_spots(start, end):
    spot = np.random.uniform(start, end)
    spots.append(spot)

    if start <= spot - 1.0:
      find_spots(start, spot - 1.0)

    if spot + 1.0 <= end:
      find_spots(spot + 1.0, end)

  find_spots(0, length)

  return len(spots) / float(length)
```

Listing A.2: Parking Problem - Recursive - standard case

In listing A.3 we see the iterative implementation of the code that parks the cars and calculates the required ratio for the standard case. We can see that there are extra functions to check if there are available parking spots on the line, to check if a spot can be occupied by a car, and to draw a parking spot uniformly from the line. The main function retrieves a parking spot, if a car fits in the spot it is parked, and iterates in this fashion until there are no more spots available for parking. A counter is maintained that prevents the iterative function from continuing indefinitely.

```
def spots_available(spots, length):
  if 0 <= spots[0] - 1.0:
    return True

  for i in range(len(spots) - 1):
    if spots[i] + 1.0 <= spots[i + 1] - 1.0:
      return True

  return spots[-1] + 1 <= length - 1.0

def spot_found(spot, spots, length):
  if 0 <= spot <= spots[0] - 1.0:
    return True

  for i in range(len(spots) - 1):
    if spots[i] + 1.0 <= spot <= spots[i + 1] - 1.0:
      return True

  return spots[-1] + 1.0 <= spot <= length - 1.0

def get_parking(length):
  return np.random.uniform(0, length)

def parking_problem(length):
  spots = [get_parking(length)]
  count = 0

  while spots_available(spots, length):
    spot = get_parking(length)

    if spot_found(spot, spots, length):
      spots.append(spot)
      spots.sort()

    count += 1

    if count == 10000000:
      break
```

```
    return len(spots) / float(length)
```

Listing A.3: Parking Problem - Iterative - standard case

In listing A.4 we see the recursive implementation of the code that parks the cars and calculates the required ratio for the discrete case.

```
def parking_problem(length):
  spots = []

  def find_spots(start, end):
    spot = np.random.randint(start, end)
    spots.append(spot)

    if start <= spot - 2:
      find_spots(start, spot - 1)

    if spot + 3 <= end:
      find_spots(spot + 2, end)

  find_spots(0, length)

  return (len(spots) * 2) / float(length)
```

Listing A.4: Parking Problem - Recursive - discrete case

In listing A.5 we see the iterative implementation of the code that parks the cars and calculates the required ratio for the discrete case. Similar to the standard case, there are extra functions provided with the difference being that the car length is set to 2, and an integer parking spot is drawn from the line.

```
def spots_available(spots, length):
  if 0 <= spots[0] - 2:
    return True

  for i in range(len(spots) - 1):
```

```
    if spots[i] + 2 <= spots[i + 1] - 2:
      return True

  return spots[-1] + 2 <= length - 1

def spot_found(spot, spots, length):
  if 0 <= spot <= spots[0] - 2:
    return True

  for i in range(len(spots) - 1):
    if spots[i] + 2 <= spot <= spots[i + 1] - 2:
      return True

  return spots[-1] + 2 <= spot <= length - 1

def get_parking(length):
  return np.random.randint(0, length)

def parking_problem(length):
  spots = [get_parking(length)]
  count = 0

  while spots_available(spots, length):
    spot = get_parking(length)

    if spot_found(spot, spots, length):
      spots.append(spot)
      spots.sort()

    count += 1

    if count == 10000000:
      break

  return (len(spots) * 2) / float(length)
```

Listing A.5: Parking Problem - Iterative - discrete case

In listing A.6 we see the recursive implementation of the code that parks the cars and calculates the required ratio for the overlap case. For the overlap case we have added an extra function for calculating the total length of the gaps. This is necessary in the overlap case precisely because cars can overlap when they park, and hence the number of cars can not be used as a measure of the coverage.

```
def parking_problem(length, overlap):
  exclusion = 1.0 - overlap
  spots     = []

  def find_spots(start, end):
    spot = np.random.uniform(start, end)
    spots.append(spot)

    if start <= spot - exclusion:
      find_spots(start, spot - exclusion)

    if spot + exclusion <= end:
      find_spots(spot + exclusion, end)

  find_spots(0, length)

  return (length - total_gaps(sorted(spots))) / float(length)

def total_gaps(spots):
  total = 0

  for i in range(len(spots) - 1):
    value = spots[i + 1] - (spots[i] + 1)
    if value > 0:
      total += value

  value = spots[-1] - (spots[-2] + 1)
  if value > 0:
    total += value

```

```
    return total
```

Listing A.6: Parking Problem - Recursive - overlap case

In listing A.7 we see the iterative implementation of the code that parks the cars and calculates the required ratio for the overlap case.

```
def spots_available(spots, length, overlap):
  exclusion = 1.0 - overlap

  if 0 <= spots[0] - exclusion:
    return True

  for i in range(len(spots) - 1):
    if spots[i] + exclusion <= spots[i + 1] - exclusion:
      return True

  return spots[-1] + exclusion <= length - exclusion

def spot_found(spot, spots, length, overlap):
  exclusion = 1.0 - overlap

  if 0 <= spot <= spots[0] - exclusion:
    return True

  for i in range(len(spots) - 1):
    if spots[i] + exclusion <= spot <= spots[i + 1] - exclusion:
      return True

  return spots[-1] + exclusion <= spot <= length - exclusion

def get_parking(length):
  return np.random.uniform(0, length)

def total_gaps(spots):
  total = 0

  for i in range(len(spots) - 1):
```

```python
    value = spots[i + 1] - (spots[i] + 1)
    if value > 0:
      total += value

  value = spots[-1] - (spots[-2] + 1)
  if value > 0:
    total += value

  return total

def parking_problem(length, overlap):
  spots = [get_parking(length)]
  count = 0

  while spots_available(spots, length, overlap):
    spot = get_parking(length)

    if spot_found(spot, spots, length, overlap):
      spots.append(spot)
      spots.sort()

    count += 1

    if count == 10000000:
      break

  return (length - total_gaps(sorted(spots))) / float(length)
```

Listing A.7: Parking Problem - Iterative - overlap case

The code is available at `https://github.com/cowboysmall/simulations` and can be cloned using the following command:

```
git clone https://github.com/cowboysmall/simulations
```

once the repository has been cloned, move into the simulations directory and run the simulations. And example of running a simulation is shown below:

```
python3 simulations/parking/parking_06.py -n 100 -l 1000 -o 0.3
```

where:

- **parking_01.py** is the standard parking problem, iterative case
- **parking_02.py** is the standard parking problem, recursive case
- **parking_03.py** is the discrete parking problem, iterative case
- **parking_04.py** is the discrete parking problem, recursive case
- **parking_05.py** is the overlap parking problem, iterative case
- **parking_06.py** is the overlap parking problem, recursive case

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